~~Reproduce the dyn. structure factor~~

$$H_{LAT} = J\sum_n \left(S^x_n S^x_{n+1} + S^y_n S^y_{n+1} + \Delta S^z_n S^z_{n+1}\right) + D\hat{z}\cdot\sum_n(\vec{S}_n\times\vec{S}_{n+1})$$

$$H_0 = \frac{2\pi v}{3}\int dx\left(\vec{J}_R\cdot\vec{J}_R + \vec{J}_L\cdot\vec{J}_L\right) \quad - \sum_n (h_x S^x_n + h_z S^z_n)$$

$$H_{bs} = -g_{bs}\int dx\left(J^x_R J^x_L + J^y_R J^y_R + (1+\lambda) J^z_R J^z_L\right)$$

$$\lambda = c\,(1-\Delta) + c'\frac{D^2}{J^2}.$$

Lattice rotation: $S^+_n = e^{i\alpha n}\,\tilde{S}^+_n \quad , \alpha = \tan^{-1}\left(\frac{D}{J}\right) \in (0,\pi/2)$

$$H_{LAT} \to \tilde{H}_{LAT} = \tilde{J}\sum_n\left(\tilde{S}^x_n\tilde{S}^x_{n+1} + \tilde{S}^y_n\tilde{S}^y_{n+1} + \Delta_{eff}\,\tilde{S}^z_n\tilde{S}^z_{n+1}\right)$$

$$- h_x\sum_n\left(\tilde{S}^+_n e^{i\alpha n} + \tilde{S}^-_n e^{-i\alpha n}\right) - h_z\sum_n \tilde{S}^z_n.$$

$$\tilde{J} = \frac{J}{\cos\alpha}\quad , \Delta_{eff} = \Delta\cos\alpha \leq \Delta.$$

$$\tan\alpha = \frac{D}{J} \Rightarrow \cos\alpha = \frac{1}{\sqrt{1+\tan^2\alpha}} = \frac{1}{\sqrt{1+D^2/J^2}}$$

$$\tilde{J} = J\sqrt{1+D^2/J^2} = \sqrt{J^2+D^2}.$$

$$\Delta_{eff} = \frac{\Delta}{\sqrt{1+D^2/J^2}} \cong \Delta\left(1-\frac{D^2}{2J^2}\right) = \left(1+(\Delta-1)\right)\left(1-\frac{D^2}{2J^2}\right)$$
$$\cong 1 - \frac{D^2}{2J^2} + \Delta - 1 =$$
$$= \Delta - \frac{D^2}{2J^2}.$$

* When $\Delta_{eff} = 1$, that is when $\Delta = \sqrt{1+D^2/J^2}$, the rotated model $\tilde{H}_{LAT}$ is isotropic.

Hence it is described by H_{bs} with $\lambda = 0$.

But $\lambda = c\left(1-\left(1+\frac{D^2}{2J^2}\right)\right) + c'\frac{D^2}{J^2} = \frac{D^2}{J^2}\left(c' - \frac{1}{2}c\right).$

$\Rightarrow$ it must be that $c' = \frac{1}{2}c$.

$\lambda = c(1-\Delta_{eff})$

$\Rightarrow$ for $\Delta = 1$ $\lambda = c\frac{D^2}{2J^2}$. More generally, $\lambda = c\left(1-\Delta+\frac{D^2}{2J^2}\right)$.

(i) XXZ model with DM: $D \neq 0$, $h_x = h_z = 0$.

$$\theta_R = \tan^{-1}\left(\frac{\tilde{D}}{h_x}\right) - \frac{\pi}{2} = 0$$

$$\theta_L = -\tan^{-1}\left(\frac{\tilde{D}}{h_x}\right) - \frac{\pi}{2} = -\pi.$$

$$\theta^{\pm} = \theta_R \pm \theta_L \Rightarrow \begin{matrix} \theta^+ = -\pi \\ \theta^- = \pi \end{matrix}$$

$$y_x(0) = -\frac{g b_s}{2\pi v}\left[\left(1+\frac{\lambda}{2}\right)\cos\theta^- - \frac{\lambda}{2}\cos\theta^+\right] = \frac{g b_s}{2\pi v}$$

(4.11) $$y_y(0) = -\frac{g b_s}{2\pi v} = -y_x(0) \;;\quad y_A = \tilde{y}_A = 0$$

$$y_e(0) = \frac{g b_s}{2\pi v} \;;\quad y_z(0) = -\frac{g b_s}{2\pi v}\left[\left(1+\frac{\lambda}{2}\right)(-1) + \frac{\lambda}{2}(-1)\right] =$$

$$= \frac{g b_s}{2\pi v}(1+\lambda).$$

$\Rightarrow$ RG (4.13):

$$\left.\begin{matrix} \dot{y}_x = y_y y_z \\ \dot{y}_y = y_x y_z \end{matrix}\right\} \quad \frac{d}{dl}(y_x - y_y) = -(y_x - y_y) y_z$$

$$\dot{y}_z = y_x y_y \qquad \frac{d}{dl}(y_x + y_y) = (y_x + y_y) y_z$$

$$\Downarrow \quad y_B = \tfrac{1}{2}(y_x + y_y)$$

$$\dot{y}_B = y_B y_z$$

$$\Rightarrow \ln\frac{y_B(l)}{y_B(0)} = \int_0^l dl\, y_z(l)$$

$$y_B(l) = y_B(0)\, e^{\int_0^l dx\, y_z(x)}$$

but $y_B(0) = 0 \Rightarrow y_B(l) = 0$

$$\Rightarrow y_x(l) = -y_y(l)$$

$$\Rightarrow \begin{cases} \dot{y}_x = -y_x y_z \\ \dot{y}_z = -y_x^2 \end{cases}$$

OR, for $\left.\begin{matrix} y_e = \frac{1}{2}(y_x - y_y) = y_x \\ y_\sigma = -y_z \end{matrix}\right\}$ $\quad \begin{matrix} \dot{y}_e = y_\sigma y_e \\ \dot{y}_\sigma = y_e^2 \end{matrix}$ (4.14)

$$y_e(0) = \frac{g b_s}{2\pi v}$$

$$y_\sigma(0) = -\frac{g b_s}{2\pi v}(1+\lambda) = -y_e(0)(1+\lambda)$$

Integral of motion: $y_\sigma^2 - y_c^2 \rightarrow \frac{d}{dl}(y_\sigma^2 - y_c^2) = 2\, y_\sigma\, y_c^2 - 2 y_c (y_\sigma y_c) = 0$

$\Rightarrow y_\sigma^2(l) = y_c^2(l) + y_\sigma^2(0) - y_c^2(0)$

Input for $\lambda > 0$ $\quad y_c(l \to \infty) \to 0$

$\boxed{K_l = \cancel{\tfrac{1}{4}}\, 1 - \frac{1}{2}\, y_\sigma(l)}$ $\quad y_\sigma(l \to \infty) \to 2\left(1 - K_{inf}\right) = 2\left[1 - \left(1 - \frac{\cos^{-1}(\Delta_{eff})}{\pi}\right)\right] =$

$= \frac{2}{\pi} \cos^{-1}(\Delta_{eff})$

Hence, for $\underline{\underline{\Delta = 1}}$, $\Delta_{eff} = \frac{1}{\sqrt{1 + D^2/J^2}} < 1$

$\Rightarrow y_\sigma(\infty) = \frac{2}{\pi} \arccos \frac{1}{\sqrt{1 + \frac{D^2}{J^2}}} = \frac{2}{\pi}\left(\frac{D}{J} - \frac{1}{3}\left(\frac{D}{J}\right)^3\right).$

But $y_\sigma^2(\infty) = \underset{\substack{\downarrow \\ 0}}{y_c^2(\infty)} + \underset{\substack{\downarrow \\ y_c(0)(1+\lambda)}}{y_\sigma^2(0)} - y_c^2(0)$

$y_\sigma^2(\infty) = y_c^2(0)\left((1+\lambda)^2 - 1\right) = y_c^2(0)\left(2\lambda + \lambda^2\right)$ [for $\lambda > 0$!]

$y_\sigma(\infty) = y_c(0)\sqrt{\lambda^2 + 2\lambda} = \frac{g_{bs}}{2\pi v}\sqrt{\lambda^2 + 2\lambda} = \frac{2}{\pi}\frac{D}{J}\left(1 - \frac{D^2}{3J^2}\right)$

$$\lambda^2 + 2\lambda = \left(\frac{2\pi v}{g_{bs}}\ \frac{2}{\pi}\right)^2 \left(\frac{D}{J}\right)^2 \left(1 - \frac{D^2}{3J^2}\right)^2.$$

But for $\underline{\underline{\Delta = 1}}$ $\quad \lambda = c\, \frac{D^2}{2J^2} \ll 1$ ($\Rightarrow$ neglect λ^2)

$$\Rightarrow 2\lambda = c\, \frac{D^2}{J^2} = \left(\frac{2\pi v}{g_{bs}}\ \frac{2}{\pi}\right)^2 \left(\frac{D}{J}\right)^2$$

$$c = \left(\frac{2}{\pi}\ \frac{2\pi v}{g_{bs}}\right)^2.$$

Eq. (B2)

(iv) XXX $(\Delta = 1)$, $h_x = D = 0$ but $h_z \neq 0$:

$$y_x(0) = y_y(0) = y_z(0) = -\frac{g_{bs}}{2\pi v}$$

$$\Rightarrow \dot{y}_a = y_a^2 \Rightarrow y_a(\ell) = \frac{y_a(0)}{1 - y_a(0)\,\ell}$$

indeed $$\dot{y}_a(\ell) = -\frac{y_a(0)}{\left(1 - y_a(0)\ell\right)^2} \cdot \left(-y_a(0)\right) = \left(\frac{y_a(0)}{1 - y_a(0)\ell}\right)^2 .$$

$$y_\sigma(\ell) = -y_z(\ell) = \frac{\frac{g_{bs}}{2\pi v}}{1 + \frac{g_{bs}}{2\pi v}\,\ell} = \frac{1}{\frac{2\pi v}{g_{bs}} + \ell}$$

Flow stops at $\ell_\phi = \ln\left(\frac{v}{h_z}\right)$

$$K(\ell_\phi) = 1 - \frac{1}{2} y_\sigma(\ell_\phi) = 1 - \frac{g_{bs}}{2 \cdot 2\pi v} \; \frac{1}{1 + \frac{g_{bs}}{2\pi v}\,\ell_\phi}$$

$$K(\ell_\phi) = 1 - \frac{1}{2\left(\frac{2\pi v}{g_{bs}} + \ell_\phi\right)} =$$

$$= 1 - \frac{1}{2\left(\ln\frac{v}{h_z} + \ln e^{\frac{2\pi v}{g_{bs}}}\right)} = 1 - \frac{1}{2\ln\left(\frac{h_0}{h_z}\right)}$$

where $h_0 = v\, e^{\frac{2\pi v}{g_{bs}}}$

So that for $g_{bs} \to 0$ (Haldane-Shastry chain) $h_0 \to \infty$ such that $K = 1$ with no log corrections!

(Note: h_0 in Eq.(B4) is incorrect)

With $g_{bs} = 0.23 \cdot 2\pi v$: $h_0 = v\, e^{\frac{1}{0.23}} = \frac{\pi}{2} e^{\frac{1}{0.23}}$ J

$$e^{-i2t_\varphi x}, \quad t_\varphi = \frac{h_z}{v}.$$

$$a_\ell = a_0 e^\ell \Rightarrow \ell = \ln\frac{a_\ell}{a_0} = \ln\left(\frac{1}{a_0}\,\frac{1}{2t_\varphi}\right) =$$

$$\Rightarrow \ell_\phi = \ln\left(\frac{v}{2h_z}\right).$$

$$y_\sigma(\ell_\phi) = \frac{1}{\ell_\phi + \frac{2\pi v}{g_{bs}}} = \frac{1}{\ln\left(\frac{v}{2h_z}\right) + \ln\left(e^{\frac{2\pi v}{g_{bs}}}\right)} =$$

$$= \frac{1}{\ln\left(\frac{v\, e^{\frac{2\pi v}{g_{bs}}}}{2h_z}\right)} \Rightarrow h_0 = \frac{1}{2} v\, e^{\frac{2\pi v}{g_{bs}}} =$$

$$= \frac{\pi}{4} J\, e^{\frac{2\pi v}{g_{bs}}} = 60\ \text{J}.$$

what's wrong?

Angular stability (p.8 of Garate-Affleck)

$\xi_c = e^{l_c}$ where $y_c(l_c) = 1$. Thus ξ_c is the correlation length of the SDW state.

With $h_z \neq 0$, $t_\theta = \frac{1}{2v}\left(\sqrt{(D+h_z)^2 + h_x^2} - \sqrt{(D-h_z)^2 + h_x^2}\right)$

$$\approx \frac{D h_z}{v\sqrt{D^2+h_x^2}} = \sqrt{\frac{D^2}{D^2+h_x^2}}\ \frac{h_z}{v} \neq 0 \text{ too}$$

$\Rightarrow \frac{1}{t_\theta}$ is the associated spatial scale of the field h_z.

If $\xi_c \ll \frac{1}{t_\theta}$, the SDW wins.

If $\xi_c \gg \frac{1}{t_\theta}$, then it is destroyed.

Garate-Affleck: if $l_c = 10$, then $\xi_c = e^{10}$

$$\Rightarrow (t_\theta)_{crit} \geqslant \xi_c^{-1} = \exp(-10).$$

$$\Rightarrow \frac{h_z}{v}\sqrt{\frac{D^2}{D^2+h_x^2}} > e^{-l_c}$$

which means that a tiny field ($h_z > v e^{-l_c}$) is enough for destroying the ordered SDW phase.

* Thus, $t_\theta \neq 0$ simply serves to cut the RG flow of y_c to strong coupling. That's all.

$$\Delta_{eff} = \frac{\Delta}{\sqrt{1+D^2/J^2}} = \underbrace{(\Delta+1-1)}_{[1+(\Delta-1)]}\left(1-\frac{D^2}{2J^2}\right) = 1+\Delta-1-\frac{D^2}{2J^2} + O\left((\Delta-1)\frac{D^2}{J^2}\right)$$

$$= 1-\left(1-\Delta+\frac{D^2}{2J^2}\right).$$

Then $\lambda = c\left(1-\Delta+\frac{D^2}{2J^2}\right) = c\left(1-\Delta_{eff}\right)$.

This is a statement on p. 1.

$\Rightarrow$ low-energy chain Hamiltonian $(\vec{D} = D\hat{z})$

$$H_0 = \frac{2\pi v}{3}\int dx \; \vec{J}_R\cdot\vec{J}_R + \vec{J}_L\cdot\vec{J}_L$$

$$H_{bs} = -g_{bs}\int dx\left(J_R^x J_L^x + J_R^y J_L^y + (1+\lambda)\, J_R^z J_L^z\right)$$

$$\lambda = c\left(1-\Delta+\frac{D^2}{2J^2}\right) = \lambda_{xc}+\lambda_{DM} \text{ of Garate-Affleck}$$

$$V = H_{field} = \int dx \left\{-h_x\left(J_R^x+J_L^x\right) - h_z\left(J_R^z+J_L^z\right) + \tilde{D}\left(J_R^z-J_L^z\right)\right\}$$

and V is handled by the chiral rotation.

Low-energy inter-chain $H' = J'\int dx \; \vec{N}_y(x)\cdot\vec{N}_{y+1}(x)$.

Analyze two situations: (A) $\vec{h} \parallel \vec{D} \Rightarrow h_x = 0,\ h_z \neq 0$

(B) $\vec{h} \perp D \Rightarrow h_x \neq 0,\ h_z = 0$.

(A) $V_A = \int dx\left(-d_R J_R^z - d_L J_L^z\right) \Rightarrow$ no rotation is required.

$$d_R = h_z - D\,,\quad d_L = h_z + D.$$

$$J^z_{\nu=R/L} = \frac{1}{2}\left(\psi^+_{\nu\uparrow}\psi_{\nu\uparrow} - \psi^+_{\nu\downarrow}\psi_{\nu\downarrow}\right) =$$

$$J_R^z = \frac{1}{2}\frac{1}{\sqrt{4\pi}}\partial_x\left(\varphi_\uparrow-\theta_\uparrow-(\varphi_\downarrow-\theta_\downarrow)\right) = \frac{1}{2\sqrt{2\pi}}\partial_x\left(\varphi_\sigma-\theta_\sigma\right)$$

$$J_L^z = \frac{1}{2\sqrt{2\pi}}\,\partial_x(\varphi_\sigma + \theta_\sigma)$$

$$\Rightarrow \quad J_R^z + J_L^z = \frac{1}{\sqrt{2\pi}}\,\partial_x \varphi_\sigma$$

$$(J_R^z - J_L^z) = -\frac{1}{\sqrt{2\pi}}\,\partial_x \theta_\sigma$$

$$V_A = \int dx \left(-h_z \frac{\partial_x \varphi_\sigma}{\sqrt{2\pi}} - D\,\frac{\partial_x \theta_\sigma}{\sqrt{2\pi}} \right)$$

$$H_0 = \frac{1}{2} v \int dx \left((\partial_x \varphi_\sigma)^2 + (\partial_x \theta_\sigma)^2 \right)$$

$$\frac{1}{2} v \left[(\partial_x \varphi_\sigma)^2 - \frac{2 h_z}{\sqrt{2\pi}\, v}\,\partial_x \varphi_\sigma \right] = \frac{1}{2} v \left(\partial_x \varphi_\sigma - \frac{h_z}{\sqrt{2\pi}\, v} \right)^2 - \frac{v}{2}\left(\frac{h_z}{\sqrt{2\pi}\, v}\right)^2$$

$$\Rightarrow \quad \underline{\varphi_\sigma = \varphi + \frac{h_z x}{\sqrt{2\pi}\, v}}$$ no σ- subindex

These shifts affect $\vec{J}_{R/L}$ and $\vec{N}$!

Similarly, $\underline{\theta_\sigma = \theta + \frac{D x}{\sqrt{2\pi}\, v}}$

$$\Rightarrow H_0 + V_A = \int dx\ \frac{1}{2} v \left\{ (\partial_x \varphi)^2 + (\partial_x \theta)^2 \right\}$$ in terms of shifted pair $\{\varphi, \theta\}$.

$$J_R^z + J_L^z = \frac{1}{\sqrt{2\pi}}\,\partial_x \varphi_\sigma = \frac{1}{\sqrt{2\pi}}\,\partial_x \varphi + \frac{h_z}{2\pi v}$$

$$J_R^z - J_L^z = \frac{-1}{\sqrt{2\pi}}\,\partial_x \theta - \frac{D}{2\pi v}.$$

$$J_R^z = \frac{1}{2}\left[\frac{\partial_x(\varphi - \theta)}{\sqrt{2\pi}} + \frac{h_z - D}{2\pi v} \right] \equiv M_R^z + \frac{h_z - D}{4\pi v}$$

$$J_L^z = \frac{1}{2}\left[\frac{\partial_x(\varphi + \theta)}{\sqrt{2\pi}} + \frac{h_z + D}{2\pi v} \right] \equiv M_L^z + \frac{h_z + D}{4\pi v}$$

$$J_R^+ = \frac{i}{4\pi a_0} e^{-i\sqrt{2\pi}(\varphi_\sigma - \theta_\sigma)} = \frac{i}{4\pi a_0} e^{-i\sqrt{2\pi}\left(\varphi - \theta + \frac{h_z - D}{\sqrt{2\pi} v} x\right)}$$

$$J_L^+ = \frac{i}{4\pi a_0} e^{i\sqrt{2\pi}(\varphi_\sigma + \theta_\sigma)} = \frac{i}{4\pi a_0} e^{i\sqrt{2\pi}\left(\varphi + \theta + \frac{h_z + D}{\sqrt{2\pi} v} x\right)}$$

$$J_R^{+} \Rightarrow M_R^+ e^{-i\frac{h_z - D}{v} x}, \quad J_R^- = M_R^- e^{i\frac{h_z - D}{v} x}$$

$$J_L^+ \Rightarrow M_L^+ e^{i\frac{h_z + D}{v} x}, \quad J_L^- = M_L^- e^{-i\frac{h_z + D}{v} x}$$

Then $J_R^x J_L^x + J_R^y J_L^y = \frac{1}{2}(J_R^+ + J_R^-)\frac{1}{2}(J_L^+ + J_L^-)$

$$- \frac{1}{4}(J_R^+ - J_R^-)(J_L^+ - J_L^-) = \frac{1}{4}(J_R^+ J_L^+ + J_R^- J_L^-$$

$$+ J_R^+ J_L^- + J_R^- J_L^+) - \frac{1}{4}(J_R^+ J_L^+ + J_R^- J_L^- - J_R^+ J_L^- - J_R^- J_L^+)$$

$$= \frac{1}{2}(J_R^+ J_L^- + J_R^- J_L^+) = \frac{1}{2}\left(M_R^+ M_L^- e^{-i\frac{2h_z x}{v}} + h.c.\right)$$

$$\Rightarrow H_{bs} = -g_{bs} \int dx \left\{ \frac{1}{2}\left(M_R^+ M_L^- e^{-i2t_\varphi x} + M_R^- M_L^+ e^{i2t_\varphi x}\right) + (1+\lambda) M_R^z M_L^z \right\} \quad t_\varphi = +\frac{h_z}{v}$$

and we neglected ~~linear~~ small terms ~

$$\int dx \, (-g_{bs})(1+\lambda)\left[\frac{h_z}{4\pi v}(M_R^z + M_L^z) + \frac{D}{4\pi v}(M_R^z - M_L^z)\right].$$

coming from linear shifts: $J_R^z J_L^z = \left(M_R^z + \frac{h_z - D}{4\pi v}\right)\left(M_L^z + \frac{h_z + D}{4\pi v}\right) =$

$$= M_R^z M_L^z + \underbrace{\frac{1}{4\pi v}\left[M_R^z (h_z + D) + M_L^z (h_z - D)\right]}_{\frac{1}{4\pi v}\left[h_z (M_R^z + M_L^z) + D(M_R^z - M_L^z)\right]}.$$

$$R_s = \frac{\eta_s}{\sqrt{2\pi a_0}} e^{i\sqrt{\pi}(\varphi_s - \theta_s)} \quad , L_s = \frac{\eta_s}{\sqrt{2\pi a_0}} e^{-i\sqrt{\pi}(\varphi_s + \theta_s)}$$

$$N^+ = R^+_\uparrow L_\downarrow + L^+_\uparrow R_\downarrow = \frac{\overbrace{(\eta_\uparrow \eta_\downarrow)}^{=i}}{2\pi a_0}\Big(e^{-i\sqrt{\pi}(\varphi_\uparrow - \theta_\uparrow)} e^{-i\sqrt{\pi}(\varphi_\downarrow + \theta_\downarrow)}$$

$$+ e^{i\sqrt{\pi}(\varphi_\uparrow + \theta_\uparrow)} e^{i\sqrt{\pi}(\varphi_\downarrow - \theta_\downarrow)} \Big) =$$

$$= \frac{i}{2\pi a_0} e^{-i\sqrt{\pi}(\sqrt{2}\varphi_\rho - \sqrt{2}\theta_\sigma)} + e^{i\sqrt{\pi}(\sqrt{2}\varphi_\rho + \sqrt{2}\theta_\sigma)}) =$$

$$= \frac{i}{2\pi a_0} \left(e^{-i\sqrt{2\pi}\varphi_\rho} + e^{i\sqrt{2\pi}\varphi_\rho} \right) e^{i\sqrt{2\pi}\theta_\sigma} = \frac{i}{\pi a_0} \cos\sqrt{2\pi}\varphi_\rho \, e^{i\sqrt{2\pi}\theta_\sigma}$$

$$\Rightarrow N^+_y(x) = \frac{\cos\sqrt{2\pi}\varphi_\rho}{\pi a_0} i e^{i\sqrt{2\pi}\theta_\sigma} \to \frac{\cos\sqrt{2\pi}\varphi_\rho}{\pi a_0}\left(i e^{i\sqrt{2\pi}\theta}\right)\left(e^{i\frac{Dx}{v}}\right)$$

$$N^z = \tfrac{1}{2}\left(R^+_\uparrow L_\uparrow + L^+_\uparrow R_\uparrow - R^+_\downarrow L_\downarrow - L^+_\downarrow R_\downarrow\right) = \tfrac{1}{2}\left(\psi^+_\uparrow \psi_\uparrow - \psi^+_\downarrow \psi_\downarrow\right).$$

$$R^+_s L_s = \frac{\eta_s \eta_s}{2\pi a_0} e^{-i\sqrt{4\pi}\phi_{Rs}} e^{-i\sqrt{4\pi}\phi_{Ls}} = \frac{1}{2\pi a_0} e^{-i\sqrt{4\pi}(\phi_{Rs}+\phi_{Ls})} \cdot e^{\frac{(-i\sqrt{4\pi})^2}{2}[\phi_{Rs},\phi_{Ls}]}$$

$$= \frac{1}{2\pi a_0} e^{-2\pi\left(\frac{i}{4}\right)} e^{-i\sqrt{4\pi}\varphi_s} = \frac{-i}{2\pi a_0} e^{-i\sqrt{4\pi}\varphi_s}$$

$$\Rightarrow N^z = \frac{1}{4\pi a_0}\left(-i e^{-i\sqrt{4\pi}\varphi_\uparrow} + i e^{i\sqrt{4\pi}\varphi_\uparrow} - (-i e^{-i\sqrt{4\pi}\varphi_\downarrow}) - i e^{i\sqrt{4\pi}\varphi_\downarrow}\right)$$

$$= \frac{i}{4\pi a_0}\left(2i \sin\sqrt{4\pi}\varphi_\uparrow - 2i\sin\sqrt{4\pi}\varphi_\downarrow\right) =$$

$$\varphi_\uparrow = \frac{1}{\sqrt{2}}(\varphi_\rho + \varphi_\sigma) \qquad = -\frac{1}{2\pi a_0}\left(\sin\sqrt{2\pi}(\varphi_\rho+\varphi_\sigma) - \sin\sqrt{2\pi}(\varphi_\rho - \varphi_\sigma)\right)$$

$$\varphi_\downarrow = \frac{1}{\sqrt{2}}(\varphi_\rho - \varphi_\sigma) \qquad = \frac{-1}{2\pi a_0} 2\cos\sqrt{2\pi}\varphi_\rho \sin\sqrt{2\pi}\varphi_\sigma$$

$$N^z_y = -\frac{\cos\sqrt{2\pi}\,\varphi_\rho}{\cancel{4}\pi a_0}\ \sin\sqrt{2\pi}\,\varphi_\sigma \Longrightarrow \frac{-\cos\sqrt{2\pi}\,\varphi_\rho}{\cancel{4}\pi a_0}\ \sin\left(\sqrt{2\pi}\,\varphi_y(x) + \frac{h_z}{v}x\right).$$

$$N^+_y(x) = \underbrace{\frac{\cos\sqrt{2\pi}\,\varphi_\rho}{\pi a_0}}_{A_3}\ i\, e^{i\sqrt{2\pi}\,\theta_y(x)}\, e^{i\frac{D_y x}{v}} \quad \left(\underline{\text{Note: } D_y = D(-1)^y!}\right)$$

$N^-_y \sim e^{-i D_y x/v}$ — Staggered DM between chains

$$\Rightarrow N^-_{y+1} = A_3\left(-i\, e^{-i\sqrt{2\pi}\,\theta_{y+1}}\right) e^{iDx/v}$$

Here $t_\theta = \frac{D}{v}$.

<u>Our limit</u> $J' \ll D$ (the opposite $J' \gg D$ limit is standard)

$\Rightarrow t' \equiv \frac{J'}{v} \ll t_\theta = \frac{D}{v} \Rightarrow$ interchain $N^+ N^-$ order is terminated by oscillating $e^{it_\theta x}$ factor.

Spatial scales: $\frac{1}{t'} \gg \frac{1}{t_\theta}$

$$H_{inter} = J'\int dx\ \frac{1}{2}\left(N^+_y N^-_{y+1} + h.c.\right) + N^z_y N^z_{y+1} \Rightarrow \text{shifts of } \varphi_\sigma, \theta_\sigma$$

say $\begin{pmatrix} y = \text{even} \\ y+1 = \text{odd}\end{pmatrix} \Rightarrow J'\int dx\ \frac{1}{2}\left(\tilde{N}^+_y e^{it_\theta x}\, \tilde{N}^-_{y+1} e^{it_\theta x} + h.c.\right) + N^z_y N^z_{y+1}$

$$N^+_y(x) = \underbrace{i A_3\, e^{i\sqrt{2\pi}\,\theta_y(x)}}_{\tilde{N}^+_y(x)}\, e^{iD_y x/v} = \tilde{N}^+_y(x)\, e^{iD_y x/v}$$

$$N^z_y(x) = i\,\frac{\cancel{A_3}}{2}\left(e^{i\sqrt{2\pi}\,\varphi_y(x) + it_\varphi x} - e^{-i\sqrt{2\pi}\,\varphi_y(x) - it_\varphi x}\right) =$$

$$\cancel{N^z_y N^z_{y+1} = -\left(\tfrac{A_3}{2}\right)^2} = \underbrace{-\frac{i}{2}A_1\, e^{-i\sqrt{2\pi}\,\varphi_y}}_{S^z_{\pi-2\delta;y}(x)}\, e^{-it_\varphi x} + \underbrace{\frac{i}{2}A_1\, e^{i\sqrt{2\pi}\,\varphi_y}}_{S^z_{\pi+2\delta;y}(x)}\, e^{it_\varphi x}$$

in notations of "extreme sensitivity" paper.

$$N^z_y N^z_{y+1} = \left(S^z_{\pi-2\delta;y}\, e^{-it_\varphi x} + S^z_{\pi+2\delta;y}\, e^{it_\varphi x}\right)\cdot\left(S^z_{\pi-2\delta;y+1}\, e^{-it_\varphi x} + S^z_{\pi+2\delta;y+1}\, e^{it_\varphi x}\right)$$

$$= S^z_{\pi-2\delta;y}\, S^z_{\pi+2\delta;y+1} + S^z_{\pi+2\delta;y}\, S^z_{\pi-2\delta;y+1}$$

$$+ S^z_{\pi-2\delta;y}\, S^z_{\pi-2\delta;y+1}\, e^{-i2t_\varphi x} + S^z_{\pi+2\delta;y}\, S^z_{\pi+2\delta;y+1}\, e^{i2t_\varphi x}$$

$$= \Big(\underbrace{e^{i\sqrt{2\pi}(\varphi_y - \varphi_{y+1})}}_{\text{not affected by } t_\varphi \text{ at all}} + \underbrace{e^{-i\sqrt{2\pi}(\varphi_y + \varphi_{y+1})}\, e^{-i2t_\varphi x}}_{\text{absent on spatial scale } > \frac{1}{t_\varphi}} + \text{h.c.} \Big)$$

<u>Spatial scales</u> $\qquad \frac{1}{t'} = \frac{v}{J'} \qquad \frac{1}{t_\theta} = \frac{v}{D} \qquad \frac{1}{t_\varphi} = \frac{v}{h_z}$

Low field h_z :	$\frac{1}{t_\varphi} \gg \frac{1}{t'} \gg \frac{1}{t_\theta}$	$h_z \ll J' \ll D$	sdw?
"Medium" field :	$\frac{1}{t'} \gg \frac{1}{t_\varphi} \gg \frac{1}{t_\theta}$	$J' \ll h_z \ll D$	sdw?
"High" field :	$\frac{1}{t'} \gg \frac{1}{t_\theta} \gg \frac{1}{t_\varphi}$	$J' \ll D \ll h_z$	cone?

<u>Low-field case</u> can be analyzed with $t_\varphi = 0$.

$$\Rightarrow \mathcal{H}_{bs} = -g_{bs}\int dx \left(M^x_R M^x_L + M^y_R M^y_L + (1+\lambda) M^z_R M^z_L \right)$$

$$= -\int dx \sum_a g_a M^a_R M^a_L .$$

$$\frac{d}{dl} g_e = -\sum_{a,b\neq e} \frac{g_a g_b}{4\pi v} \qquad \dot g_x = -\frac{1}{4\pi v}(g_y g_z + g_z g_y) = \frac{-g_y g_z}{2\pi v}$$

$$\dot g_y = -\frac{g_x g_z}{2\pi v}$$

Clearly $g_x = g_y$

$$\dot g_z = -\frac{g_x g_y}{2\pi v}$$

$$\frac{d}{dl}(g_x+g_y) = -(g_x+g_y)\frac{g_z}{2\pi v} \Rightarrow \ln(g_x+g_y)\Big|_0^l = -\int_0^l dl\,\frac{g_z}{2\pi v}$$

$$\frac{d}{dl}(g_x-g_y) = (g_x-g_y)\frac{g_z}{2\pi v} \Rightarrow \frac{(g_x-g_y)_l}{(g_x-g_y)_0} = \exp\int_0^l dl\,\frac{g_z}{2\pi v}$$

$$\Rightarrow g_x(l) = g_y(l) \text{ if } g_x(0)=g_y(0).$$

$$\Rightarrow \cancel{\dot{g} = -\frac{gg}{2\pi}} \quad \begin{cases} y = \frac{g_x}{2\pi v} = \frac{g_y}{2\pi v} \\ y_z = \frac{g_z}{2\pi v} \end{cases}$$

$$\begin{cases} \dot{y} = -y y_z \\ \dot{y}_z = -y^2 \end{cases} \Rightarrow \frac{d}{dl}(y^2-y_z^2) = 2y(-y y_z) - 2y_z(-y^2) = 0.$$

$$y^2(l) - y_z^2(l) = y^2(0) - y_z^2(0) = \eta^2(1-(1+\lambda)^2) = -C, \quad C>0 \text{ for } \lambda>0$$

$$y(0) = \frac{g_{bs}}{2\pi v} \equiv \eta, \quad y_z(0) = \frac{g_{bs}(1+\lambda)}{2\pi v} = \eta(1+\lambda)$$

$$\dot{y}_z = +(C-y_z^2) \Rightarrow \int_{y_z(0)}^{y_z(l)} \frac{dy_z}{y_z^2-C} = -l$$

$$C = \eta^2\left[(1+\lambda)^2-1\right] = y_z^2(0) - y^2(0)$$

$$\Rightarrow y_z^2(0) - C = y^2(0) > 0.$$

$$\Rightarrow \int dy\left(\frac{1}{y-\sqrt{C}} - \frac{1}{y+\sqrt{C}}\right)\frac{1}{2\sqrt{C}} = \frac{1}{2\sqrt{C}}\ln\frac{y_z-\sqrt{C}}{y_z+\sqrt{C}}$$

$$\Rightarrow \ln\frac{y_z(l)-\sqrt{C}}{y_z(0)-\sqrt{C}}\,\frac{y_z(0)+\sqrt{C}}{y_z(l)+\sqrt{C}} = -2\sqrt{C}\,l$$

$$\frac{y_z(\ell)-\sqrt{c}}{y_z(\ell)+\sqrt{c}} = \frac{y_z(0)-\sqrt{c}}{y_z(0)+\sqrt{c}}\, e^{-2\sqrt{c}\,\ell} \xrightarrow[\ell\to\infty]{} 0 \Rightarrow y_z(\ell)\to\sqrt{c}$$

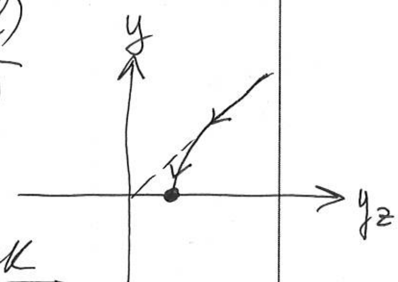
$$y_z(\ell)-\sqrt{c} = A\left(y_z(\ell)+\sqrt{c}\right)$$

$$y_z(\ell)\left(1-A\right) = \sqrt{c}\left(1+A\right)$$

$$y_z(\ell) = \sqrt{c}\,\frac{1+A}{1-A} = \sqrt{c}\,\frac{y_z(0)+\sqrt{c}+\left(y_z(0)-\sqrt{c}\right)e^{-2\sqrt{c}\ell}}{y_z(0)+\sqrt{c}-\left(y_z(0)-\sqrt{c}\right)e^{-2\sqrt{c}\ell}} =$$

$$= \sqrt{c}\,\frac{y_z(0)\left(1+e^{-2\sqrt{c}\ell}\right)+\sqrt{c}\left(1-e^{-2\sqrt{c}\ell}\right)}{y_z(0)\left(1-e^{-2\sqrt{c}\ell}\right)+\sqrt{c}\left(1+e^{-2\sqrt{c}\ell}\right)} =$$

$$= \sqrt{c}\,\frac{y_z(0)\,\mathrm{ch}(\sqrt{c}\ell)+\sqrt{c}\,\mathrm{sh}(\sqrt{c}\ell)}{y_z(0)\,\mathrm{sh}(\sqrt{c}\ell)+\sqrt{c}\,\mathrm{ch}(\sqrt{c}\ell)}$$



$$\Rightarrow y^2(\ell) \Rightarrow y_z^2(\ell)-c \xrightarrow[\ell\to\infty]{} 0 .$$

<u>OK</u>

For $\lambda<0$ the RG flow will lead to Ising order in a single chain in absence of any h_z.

$$Z = \int e^{S_0 - \int d\tau \,(H_{bs} + H_{inter})} = \int e^{S_0} \Big\{ 1 - \int d\tau \,(H_{bs} + H_{inter})$$
$$+ \frac{1}{2} \int d\tau\, d\tau' \left(H_{bs}(\tau') + H_{inter}(\tau')\right)\left(H_{bs}(\tau) + H_{inter}(\tau)\right) \Big\}$$

Need to fuse $\quad g_a \, M^a_{R\,y}(x,\tau)\, M^a_{L\,y}(x,\tau)\; N^b_y(0,0)\, N^b_{y+1}(0,0)$

$$\sum_a M^a_R\, M^a_L\, N^b = M^a_R \frac{i\varepsilon^{abc} N^c + i\delta^{ab}\varepsilon}{4\pi(v\tau + ix)} =$$

$$= \sum_a \frac{i\varepsilon^{abc}\left(i\varepsilon^{acd} N^d + i\delta^{ac}\varepsilon\right) + i\delta^{ab}\left(iN^a\right)}{(4\pi)^2 (v\tau + ix)(v\tau - ix)} =$$

$$= -\sum_a \frac{\varepsilon^{abc}\varepsilon^{acd} N^d + \delta^{ab} N^a}{(4\pi)^2\left((v\tau)^2 + x^2\right)} = \frac{-2\, N^b}{(4\pi)^2\left[(v\tau)^2 + x^2\right]}$$

$$-\varepsilon^{abc}\varepsilon^{ade} = -\delta^{bd}$$

$$\varepsilon^{abc}\varepsilon^{adf} = \delta^{bd}\delta^{cf} - \delta^{bf}\delta^{cd}$$

$$M^x_R M^x_L\, N^z_y = M^x_R \frac{i\varepsilon^{xzy} N^y}{4\pi \bar{z}} = \frac{i\varepsilon^{xzy}\, i\varepsilon^{xyz} N^z}{(4\pi\bar{z})(4\pi z)} = \frac{N^z}{(4\pi)^2 |z|^2}$$

$$M^y_R M^y_L\, N^z_y = M^y_R \frac{i\varepsilon^{yzx} N^x_y}{4\pi \bar{z}} = \frac{i\varepsilon^{yzx}\, i\varepsilon^{yxz} N^z_y}{(4\pi\bar{z})(4\pi z)} = \frac{N^z}{(4\pi)^2 |z|^2}$$

$$M^z_R M^z_L\, N^z_y = M^z_R \frac{i\varepsilon}{4\pi \bar{z}} = \frac{i(iN^z)}{(4\pi\bar{z})(4\pi z)} = -\frac{N^z}{(4\pi)^2 |z|^2}$$

$$\Rightarrow \sum_a g_a\, M^a_R M^a_L\, N^z_y N^z_{y+1} = \frac{(g_x + g_y - g_z)\, N^z_y N^z_{y+1}}{(4\pi)^2\, |z|^2}.$$

$\Rightarrow$ correction is ($\frac{1}{2}$ gone due to $\tau'.\tau$ and $\tau.\tau'$ equal terms)

$$\int d\tau d\tau' \int dx dx' \, H_{bs}(x,\tau)\, H_{inter}(x',\tau') =$$

from overall sign of H_{bs}

$$= \int dx d\tau \underbrace{\int dx' d\tau' \frac{(-)(2)}{(4\pi)^2\left[(v\tau')^2 + x'^2\right]}}_{\frac{(-4\pi)}{(4\pi)^2}\int \frac{dr\, r}{v r^2} = -\frac{dl}{4\pi v}}$$

(2) from bs acting on chains $y, y+1$

$$\cdot \left\{ (g_x + g_y - g_z)\, N^z_y N^z_{y+1} + (g_x + g_z - g_y)\, N^y_y N^y_{y+1} + (g_z + g_y - g_x)\, N^x_y N^x_{y+1} \right\}$$

$\Rightarrow$ correction to interchain term $N^z_y N^z_{y+1}$

$$\left\{ J' + J'\left(\frac{g_x + g_y - g_z}{4\pi v}\, dl\right)\right\} N^z_y N^z_{y+1}$$

$$\frac{d G_z}{dl} = G_z \underbrace{\left(1 + \tfrac{1}{2}(y_x + y_y - y_z)\right)}_{\approx\, 1 + \frac{\eta(1-\lambda)}{2}} \qquad G_z^{(0)} = J' \qquad y_j \equiv \frac{g_j}{2\pi v}$$

$$\frac{d G_x}{dl} = \frac{d G_y}{dl} = G_x \underbrace{\left(1 + \tfrac{1}{2}(y_y + y_z - y_x)\right)}_{}.$$

$$\text{approx.} \quad 1 + \frac{1}{2}\left(\eta + \eta(1+\lambda) - \eta\right) = 1 + \frac{\eta(1+\lambda)}{2}$$

But $G_{x,y}$, even grow faster for $\lambda > 0$, are wiped out at scale $l_d = \ln\left(\frac{v}{D}\right)$ [when they are still small], while slower-growing G_z is not affected! $\Rightarrow$ G_z order (SDW).

* Same behavior holds in "Intermediate field" regime ($J' \ll h_z \ll D$) since flow of BS terms is not affected by $l_d = \ln\left(\frac{v}{D}\right)$ when $N_Y^+ N_{Y+1}^-$ get removed.

* "High field" may be different because

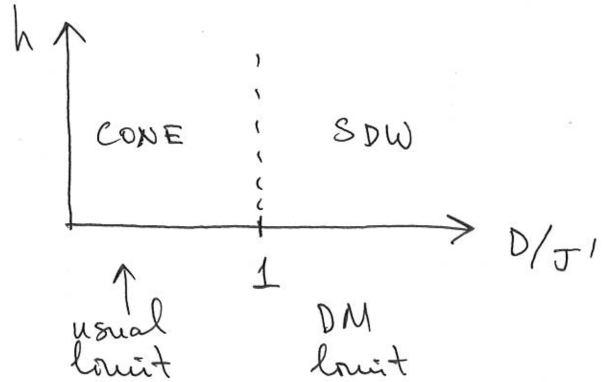
$$H_{bs} \rightarrow \int dx \, (-g_z) \, M_R^z M_L^z \quad \text{only}$$

(can obtain by setting $g_{x,y} = 0$ in RG eqns)

$$\left.\begin{aligned} \frac{dG_z}{dl} &= G_z \left(1 - \tfrac{1}{2} y_z\right) \\ \frac{dG_{x,y}}{dl} &= G_{x,y} \left(1 + \tfrac{1}{2} y_z\right) \end{aligned}\right\}$$

for $l_\psi < l < l_d$.
for $l > l_d$, where G is still small, $G_{x,y}$ is effectively removed and again G_z survives.

Thus SDW along z (DM axis) seems to be the GS for as long as $D \gg J'$, so that $\tilde{N}^+ \hat{N}^- e^{i 2 D x / v}$ can not stabilize.



Conversely, we expect cone state for $J' > D$: in this limit $e^{i 2 D x / v}$ factor does not develop oscillations by the time $G_{x,y}$ couplings become strong.

(B) $h_z = 0$, $h_x = h \neq 0$. Back to p.7 and do the chiral rotation (the same way as Garate-Affleck, GA, did it)

$$V = \int dx \left(-h_x J_R^x + D J_R^z - h_x J_L^x - D J_L^z \right)$$

$$\begin{pmatrix} J^x \\ J^y \\ J^z \end{pmatrix}_{R/L} = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}_{R/L} \begin{pmatrix} M^x \\ M^y \\ M^z \end{pmatrix}_{R/L}$$

Choosing $\begin{cases} \theta_R = -\frac{\pi}{2} + \tan^{-1}\left(\frac{d}{h_x}\right) \\ \theta_L = -\frac{\pi}{2} - \tan^{-1}\left(\frac{d}{h_x}\right) \end{cases}$, $\theta_L = -\theta_R - \pi$

gives $V = \int dx \sqrt{h_x^2 + D^2}\, (M_R^z + M_L^z)$. Note $M_R^z + M_L^z = \frac{\partial_x \varphi_\sigma}{\sqrt{2\pi}}$.

$\begin{cases} \psi_{R,s} = e^{-i\theta_R \sigma^y/2} R_s \\ \psi_{L,s} = e^{-i\theta_L \sigma^y/2} L_s \end{cases}$ $\psi_{R/L}$ = original fermions, R/L = rotated ones

$$\vec{J}_R = \psi^+_{R,s} \left(\frac{1}{2}\vec{\sigma}\right) \psi_{R,s'} = R_s^+ e^{i\theta_R \sigma^y/2} \left(\frac{\vec{\sigma}}{2}\right) e^{-i\theta_R \sigma^y/2} R_{s'}$$

$$e^{i\theta_R \sigma^y/2} = \cos(\theta_R/2) + i\sigma^y \sin(\theta_R/2)$$

$$\vec{J}_R = \frac{1}{2} R_s^+ \left\{ \begin{pmatrix} \sin\theta, \cos\theta \\ \cos\theta, -\sin\theta \end{pmatrix}, \sigma^y, \begin{pmatrix} \cos\theta & -\sin\theta \\ -\sin\theta & -\cos\theta \end{pmatrix} \right\} R_{s'} =$$

$$\cancel{\underbrace{\sin\theta}_{\frac{e^{i\theta}-e^{-i\theta}}{2i}} \sigma^z + \underbrace{\cos\theta}_{\frac{e^{i\theta}+e^{-i\theta}}{2}} \sigma^x = \frac{1}{2} e^{i\theta}(\sigma^x - i\sigma^z) + e^{-i\theta}(\sigma^x + i\sigma^z)}$$

$$= \frac{1}{2} R_s^+ \left\{ (\cos\theta\, \hat{\sigma}^x + \sin\theta\, \hat{\sigma}^z), \hat{\sigma}^y, (-\sin\theta\, \hat{\sigma}^x + \cos\theta\, \hat{\sigma}^z) \right\} R_{s'}$$

$$\Rightarrow \vec{J}_R = \left(\cos\theta\, M_R^x + \sin\theta\, M_R^z,\ M_R^y,\ -\sin\theta\, M_R^x + \cos\theta\, M_R^z\right)$$

$$= \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \vec{M}_R \qquad \underline{\text{OK}}$$

$$\vec{S} = \left(\psi_R^+ e^{-ik_F x} + \psi_L^+ e^{+ik_F x}\right) \frac{\vec{\sigma}}{2} \left(\psi_R e^{ik_F x} + \psi_L e^{-ik_F x}\right)$$

$$\vec{S}_{2k_F} = \frac{1}{2}\left(\psi_R^+ \vec{\sigma} \psi_L e^{-i2k_F x} + \psi_L^+ \vec{\sigma} \psi_R e^{i2k_F x}\right)$$

$2k_F = \pi$ $$\vec{N} = \frac{1}{2}\left(\psi_R^+ \vec{\sigma} \psi_L + \psi_L^+ \vec{\sigma} \psi_R\right) =$$

$$= \frac{1}{2}\Big(R_s^+ e^{i\theta_R \sigma^y/2} \vec{\sigma} e^{-i\theta_L \sigma^y/2} L_{s'} +$$

$$+ L_s^+ e^{i\theta_L \sigma^y/2} \vec{\sigma} e^{i\theta_R \sigma^y/2} R_{s'}\Big).$$

I find (file Garate_Affleck.nb) that

$$N^x = +\frac{1}{2}\left(R_s^+ (-\hat{\sigma}^z) L_{s'} + L_s^+ (-\hat{\sigma}^z) R_{s'}\right) = -\tilde{N}^z$$

$$N^z = \frac{1}{2}\left(R_s^+ (\hat{\sigma}^x) L_{s'} + L_s^+ (\hat{\sigma}^x) R_{s'}\right) = \tilde{N}^x$$

$$N^y = \frac{i}{2}\left\{R_s^+ \begin{pmatrix} \frac{D}{\sqrt{h^2+D^2}}, & \frac{-h}{\sqrt{h^2+D^2}} \\ \frac{h}{\sqrt{h^2+D^2}}, & \frac{D}{\sqrt{h^2+D^2}} \end{pmatrix} L_{s'} + L_s^+ \begin{pmatrix} D \to -D \end{pmatrix} R_{s'}\right\}$$

$$= \frac{1}{2}\left\{R_s^+ \left(\frac{iD}{\sqrt{h^2+D^2}} \hat{\sigma}^0 + \frac{h}{\sqrt{h^2+D^2}} \hat{\sigma}^y\right) L_{s'}\right.$$

$$\left. + L_s^+ \left(\frac{-iD}{\sqrt{h^2+D^2}} \hat{\sigma}^0 + \frac{h}{\sqrt{h^2+D^2}} \hat{\sigma}^y\right) R_{s'}\right\}$$

Thus $N^y = \frac{i}{2} \frac{D}{\sqrt{D^2+h^2}} \sum_s \left(R_s^+ L_s - L_s^+ R_s\right) + \frac{h}{\sqrt{h^2+D^2}} \tilde{N}^y$

p.10 $R_s^+ L_s = \frac{-i}{2\pi a_0} e^{-i\sqrt{4\pi}\varphi_s}$

$$i\left(R_s^+ L_s - L_s^+ R_s\right) = i\left(\frac{-i}{2\pi a_0} e^{-i\sqrt{4\pi}\varphi_s} - \frac{i}{2\pi a_0} e^{i\sqrt{4\pi}\varphi_s}\right) =$$

$$= \frac{1}{\pi a_0} \cos\sqrt{4\pi}\varphi_s$$

$$\Rightarrow \frac{i}{2} \sum_s \left(R_s^+ L_s - L_s^+ R_s\right) = \frac{1}{2\pi a_0}\left(\cos\sqrt{4\pi}\varphi_\uparrow + \cos\sqrt{4\pi}\varphi_\downarrow\right) =$$

$$\varphi_\uparrow = \frac{\varphi_\rho + \varphi_\sigma}{\sqrt{2}}$$

$$\varphi_\downarrow = \frac{\varphi_\rho - \varphi_\sigma}{\sqrt{2}}$$

$$= \frac{\cos\sqrt{2\pi}\varphi_\rho}{\pi a_0} \cos\sqrt{2\pi}\varphi_\sigma = A_3 \cos\sqrt{2\pi}\varphi_\sigma = \mathcal{E} = \text{dimerization}$$

$$N_y^y = \frac{D}{\sqrt{D^2+h^2}} \tilde{\mathcal{E}} + \frac{h}{\sqrt{h^2+D^2}} \tilde{N}^y = \cos\theta_R\, \tilde{\mathcal{E}} + \sin\theta_R\, \tilde{N}^y$$

y = even chain

Remember: $N^+ = A_3\, i\, e^{i\sqrt{2\pi}\theta_\sigma} \Rightarrow \begin{cases} \tilde{N}^x = -A_3 \sin\sqrt{2\pi}\theta_\sigma \\ \tilde{N}^y = A_3 \cos\sqrt{2\pi}\theta_\sigma \\ \tilde{N}^z = -A_3 \sin\sqrt{2\pi}\varphi_\sigma \\ \tilde{\mathcal{E}} = A_3 \cos\sqrt{2\pi}\varphi_\sigma \end{cases}$

odd chain

$D \to -D$

$$N_{y+1}^y = -\frac{D}{\sqrt{D^2+h^2}} \tilde{\mathcal{E}} + \frac{h}{\sqrt{h^2+D^2}} \tilde{N}^y = -\cos\theta_R\, \tilde{\mathcal{E}} - \sin\theta_R\, \tilde{N}^y$$

$$\cos\theta_R = \frac{d}{\sqrt{d^2+h^2}}, \quad \sin\theta_R = -\frac{h}{\sqrt{d^2+h^2}}$$

$$\begin{cases} \cos\frac{\theta^-}{2} = \frac{h}{\sqrt{d^2+h^2}} \\ \sin\frac{\theta^-}{2} = \frac{d}{\sqrt{d^2+h^2}} \end{cases}$$

Note that here $\theta_L = -\theta_R - \pi \Rightarrow \theta^+ = \theta_R + \theta_L = -\pi$

$$\theta^- = \theta_R - \theta_L = 2\theta_R + \pi = 2\left(\theta_R + \frac{\pi}{2}\right) = 2\tan^{-1}\left(\frac{d}{h_x}\right).$$

$\Rightarrow$ after the chiral rotation

$$H_0 = \int dx \; \frac{v}{2}\left[(\partial_x \varphi_\sigma)^2 + (\partial_x \theta_\sigma)^2\right] \quad . \text{ Of course also } \mathcal{H}_{bs} = \ldots$$

$$V = \int dx \sqrt{h^2+d^2}\,(M_R^z + M_L^z) = \int dx \frac{\sqrt{h^2+d^2}}{\sqrt{2\pi}}\, \partial_x \varphi_\sigma$$

but
$$H_{inter} = \sum_y J' \int dx \; \vec{N}_y \cdot \vec{N}_{y+1} = \sum_y J' \int dx \Big(\tilde{N}_y^z \tilde{N}_{y+1}^z + \tilde{N}_y^x \tilde{N}_{y+1}^x$$
$$+ \left(\cos\theta_R\, \tilde{\varepsilon}_y - \sin\theta_R\, \tilde{N}_y^y\right)\left(-\cos\theta_R\, \tilde{\varepsilon}_{y+1} - \sin\theta_R\, \tilde{N}_{y+1}^y\right)\Big).$$

* Now absorb V via the shift $\varphi_\sigma = \varphi - \tilde{q}_0 x \Rightarrow \partial_x \varphi_\sigma = \partial_x \varphi - q_0$

$$\int dx \; \frac{v}{2}(\partial_x \varphi_\sigma)^2 + \frac{\sqrt{h^2+d^2}}{\sqrt{2\pi}}\, \partial_x \varphi_\sigma =$$

$$= \int dx \; \frac{v}{2}(\partial_x \varphi - q_0)^2 + \sqrt{\frac{h^2+d^2}{2\pi}}\,(\partial_x \varphi - q_0) =$$

$$= \int dx \; \frac{v}{2}\left((\partial_x \varphi)^2 + q_0^2\right) + \left(-v q_0 + \frac{\sqrt{h^2+d^2}}{\sqrt{2\pi}}\right)\partial_x \varphi$$

$$\tilde{q}_0 = \frac{\sqrt{h^2+d^2}}{\sqrt{2\pi}\, v}$$
. The shift is the same for all chains.

Let $\sqrt{2\pi}\,\tilde{q}_0 = q_0 = \dfrac{\sqrt{h^2+d^2}}{v}$.

Under the shift, $\tilde{N}^{x,y}$ are not affected, while

$$\tilde{\varepsilon} \to \tilde{\varepsilon}' = A_3 \cos\left(\sqrt{2\pi}\,\varphi - q_0 x\right)$$

$$\tilde{N}^z \to \tilde{N}^{z\prime} = -A_3 \sin\left(\sqrt{2\pi}\,\varphi - q_0 x\right).$$

* In the $D \gg J'$ limit, the q_0-oscillation is reached <u>before</u> the strong coupling in interchain J' $\Rightarrow$ should drop all oscillating terms with $e^{i q_0 x}$.

$$\tilde N^z_y \tilde N^z_{y+1} - \cos^2\theta_R\ \tilde\varepsilon_y \tilde\varepsilon_{y+1} = A_3^2 \Big\{ \sin(\sqrt{2\pi}\,\varphi_y - q_0 x)\sin(\sqrt{2\pi}\,\varphi_{y+1} - q_0 x)$$
$$- \cos^2\theta_R \underbrace{\cos(\sqrt{2\pi}\,\varphi_y - q_0 x)\cos\sqrt{2\pi}(\varphi_{y+1} - q_0 x)}_{\frac{1}{2}\cos\sqrt{2\pi}(\varphi_y - \varphi_{y+1})} \Big\}$$

$$\Rightarrow \underbrace{\frac{1}{2} A_3^2 (1-\cos^2\theta_R)}_{\frac{1}{2}A_3^2 \frac{h^2}{h^2+D^2}} \cos\sqrt{2\pi}(\varphi_y - \varphi_{y+1}) + (\text{"}2q_0x\text{" oscillating terms})$$

$$= \frac{1}{2}\sin^2\theta_R\,(N^z_y N^z_{y+1} + \varepsilon_y\varepsilon_{y+1})$$ where I defined $\begin{cases} \varepsilon_y = A_3 \cos\sqrt{2\pi}\,\varphi \\ N^z_y = -A_3\sin\sqrt{2\pi}\,\varphi \end{cases}$

here φ is the field __after__ the shift

There remains $\tilde N^x_y \tilde N^x_{y+1} + \sin^2\theta_R\,\tilde N^y_y \tilde N^y_{y+1} =$

$$= A_3^2\left(\sin\sqrt{2\pi}\,\theta_y \sin\sqrt{2\pi}\,\theta_{y+1} + \sin^2\theta_R \cos\sqrt{2\pi}\,\theta_y \cos\sqrt{2\pi}\,\theta_{y+1}\right)$$

$$= \frac{1}{2}A_3^2\Big(\cos\sqrt{2\pi}(\theta_y - \theta_{y+1}) - \cos\sqrt{2\pi}(\theta_y+\theta_{y+1}) + \sin^2\theta_R\big[\cos\sqrt{2\pi}(\theta_y+\theta_{y+1})$$
$$+ \cos\sqrt{2\pi}(\theta_y - \theta_{y+1})\big]\Big) =$$

$$= \frac{1}{2}A_3^2\Big\{(1+\sin^2\theta_R)\cos\sqrt{2\pi}(\theta_y-\theta_{y+1}) - (1-\sin^2\theta_R)\cos\sqrt{2\pi}(\theta_y+\theta_{y+1})\Big\}$$

Also The terms ~~[illegible]~~ $\cos\theta_R\sin\theta_R\left(\tilde N^y_y \tilde\varepsilon_{y+1} - \tilde\varepsilon_y \tilde N^y_{y+1}\right)$ contain e^{iq_0x} factor, via $\tilde\varepsilon$ field, and thus average to zero.

Thus __after__ integrating H_{inter} up to scale $\frac{1}{q_0}$ we have:

drop tildas

$$H'_{inter} = \sum_y J'\left(\frac{v}{D}\right)\int dx\,\Big\{\overbrace{N^x_y N^x_{y+1}}^{G_x} + \overbrace{\sin^2\theta_R\,N^y_y N^y_{y+1}}^{G_y} + \frac{1}{2}\underbrace{\sin^2\theta_R\,(N^z_y N^z_{y+1} + \varepsilon_y\varepsilon_{y+1})}_{G_{\varepsilon,z}}\Big\}$$

$\sin^2\theta_R = \frac{h^2}{h^2+d^2}$

need to do the same with H_{bs}.

from $\dot G \cong G$

$G(l_d) = G(0)e^{l_d}$, $e^{l_d} = \frac{v}{d}$. Since $\frac{J'}{d} \ll 1$, $J'\left(\frac{v}{d}\right) \ll v$ still.

$$H_{bs} = -g_{bs}\int dx\ \vec{J}_R\cdot\vec{J}_L = -g_{bs}\int dx\ \left(\hat{R}(\theta_R)\vec{M}_R\right)^T\cdot\hat{R}(\theta_L)\vec{M}_L =$$

$$= -g_{bs}\int dx\ \vec{M}_R\ \hat{R}^T(\theta_R)\ \hat{R}(\theta_L)\ \vec{M}_L =$$

$$= -g_{bs}\int dx\ (M_R^x, M_R^y, M_R^z)\ \frac{\begin{pmatrix} h^2-d^2 & 0 & -2hd \\ 0 & h^2+d^2 & 0 \\ 2hd & & h^2-d^2\end{pmatrix}}{d^2+h^2}\begin{pmatrix} M_L^x \\ M_L^y \\ M_L^z\end{pmatrix} =$$

$$= -\frac{g_{bs}}{h^2+d^2}\int dx\ (M_R^x\ M_R^y\ M_R^z)\begin{pmatrix}(h^2-d^2)M_L^x - 2hd\,M_L^z \\ (h^2+d^2)M_L^y \\ 2hd\,M_L^x + (h^2-d^2)M_L^z\end{pmatrix}$$

$$= -\frac{g_{bs}}{h^2+d^2}\int dx\left\{(h^2-d^2)(M_R^x M_L^x + M_R^z M_L^z) + 2hd(M_R^z M_L^x - M_R^x M_L^z)\right.$$
$$\left. + (h^2+d^2)M_R^y M_L^y\right\}.$$

But $M_R^+ = \frac{i}{4\pi a_0}e^{-i\sqrt{2\pi}(\varphi_\sigma-\theta_\sigma)} \to \frac{i}{4\pi a_0}e^{-i\sqrt{2\pi}(\varphi-\theta_\sigma)}e^{iq_0x}$

$M_L^+ = \frac{i}{4\pi a_0}e^{i\sqrt{2\pi}(\varphi_\sigma+\theta_\sigma)} \to \frac{i}{4\pi a_0}e^{i\sqrt{2\pi}(\varphi+\theta_\sigma)}e^{-iq_0x}$

$$\sqrt{2\pi}\,\varphi_\sigma \to \sqrt{2\pi}\,\varphi - q_0x$$

i.e. $\begin{cases} M_R^+ = \tilde{M}_R^+ e^{iq_0x} \\ M_L^+ = \tilde{M}_L^+ e^{-iq_0x}\end{cases}$

$$\Rightarrow M_R^x + iM_R^y = (\tilde{M}_R^x + i\tilde{M}_R^y)\,e^{iq_0x}$$

$$M_R^x = \tilde{M}_R^x\cos q_0x - \tilde{M}_R^y\sin q_0x$$
$$M_L^x = \tilde{M}_L^x\cos q_0x + \tilde{M}_L^y\sin q_0x$$

$$M_R^y = \tilde{M}_R^y\cos q_0x + \tilde{M}_R^x\sin q_0x$$
$$M_L^y = \tilde{M}_L^y\cos q_0x - \tilde{M}_L^x\sin q_0x$$

$$\Rightarrow M^x_R M^x_L = \left(\widetilde{M}^x_R \cos q_0 x - \widetilde{M}^y_R \sin q_0 x\right)\left(\widetilde{M}^x_L \cos q_0 x + \widetilde{M}^y_L \sin q_0 x\right)$$

$$= \widetilde{M}^x_R \widetilde{M}^x_L \cos^2 q_0 x - \widetilde{M}^y_R \widetilde{M}^y_L \sin^2 q_0 x + \cos q_0 x \sin q_0 x \left(\widetilde{M}^x_R \widetilde{M}^y_L - \widetilde{M}^y_R \widetilde{M}^x_L\right)$$

$$\Rightarrow \frac{1}{2}\left(\widetilde{M}^x_R \widetilde{M}^x_L - \widetilde{M}^y_R \widetilde{M}^y_L\right)$$ after integrating over x up to $\frac{1}{q_0}$.

$$M^y_R M^y_L = \widetilde{M}^y_R \widetilde{M}^y_L \cos^2 q_0 x - \widetilde{M}^x_R \widetilde{M}^x_L \sin^2 q_0 x + \cos q_0 x \sin q_0 x (\ldots).$$

$$\Rightarrow \frac{1}{2}\left(\widetilde{M}^y_R \widetilde{M}^y_L - \widetilde{M}^x_R \widetilde{M}^x_L\right)$$

$$H_{bs} = -\frac{g_{bs}}{h^2+d^2}\int dx \Big[(h^2-d^2)\left\{\frac{1}{2}\left(\widetilde{M}^x_R \widetilde{M}^x_L - \widetilde{M}^y_R \widetilde{M}^y_L\right) + M^z_R M^z_L\right\} + \frac{1}{2}\left(\widetilde{M}^y_R \widetilde{M}^y_L - \widetilde{M}^x_R \widetilde{M}^x_L\right)(h^2+d^2)$$

$$= -\frac{g_{bs}}{h^2+d^2}\int dx\, (h^2-d^2) M^z_R M^z_L + \frac{1}{2}\left(h^2-d^2-(h^2+d^2)\right)\cdot \left(\widetilde{M}^x_R \widetilde{M}^x_L - \widetilde{M}^y_R \widetilde{M}^y_L\right)$$

$$= -g_{bs}\int dx \left\{\frac{h^2-d^2}{h^2+d^2} M^z_R M^z_L - \frac{d^2}{h^2+d^2}\underbrace{\left(\widetilde{M}^x_R \widetilde{M}^x_L - \widetilde{M}^y_R \widetilde{M}^y_L\right)}_{\frac{1}{2}\left(\widetilde{M}^+_R \widetilde{M}^+_L + h.c.\right)}\right\}$$

$$M^x_R M^x_L - M^y_R M^y_L = \frac{1}{4}\left(M^+_R + M^-_R\right)\left(M^+_L + M^-_L\right) - \frac{1}{(2i)^2}\left(M^+_R - M^-_R\right)\cdot\left(M^+_L - M^-_L\right)$$

$$= \frac{1}{4}\Big(M^+_R M^+_L + M^-_R M^-_L + M^+_R M^-_L + M^-_R M^+_L + M^+_R M^+_L + M^-_R M^-_L - M^+_R M^-_L + M^-_R M^+_L\Big)$$

$$= \frac{1}{2}\left(M^+_R M^+_L + M^-_R M^-_L\right).$$

$$\frac{h^2-d^2}{h^2+d^2} = \sin^2\theta_R - \cos^2\theta_R = 1-2\cos^2\theta_R = -\cos 2\theta_R = \cos\theta^-$$

$$\frac{d^2}{h^2+d^2} = \cos^2\theta_R = \frac{1}{2}(\cos 2\theta_R + 1) = \frac{1}{2}(1-\cos\theta^-)$$

$$2\theta_R = \theta^- - \pi$$

$$H_{bs} = -g_{bs}\int dx \left\{ \cos\theta^- M_R^z M_L^z + \frac{\cos\theta^- - 1}{4}\left(\tilde{M}_R^+ \tilde{M}_L^+ + \tilde{M}_R^- \tilde{M}_L^-\right)\right\}$$

~~For $\vec{h} \perp \vec{D}$ GA has: $\theta^+ = -\pi$, $\theta^- = 2\tan^{-1}\left(\frac{d}{h}\right)$.~~

~~$\Rightarrow$ coeff. of $M_R^z M_L^z$ is~~ $(-2\pi v y_\sigma) = g_{bs}\left[\left(1+\frac{\lambda}{2}\right)\cos\theta^- - \frac{\lambda}{2}\right]$

But H_{bs} has additional term:

$$-g_{bs}\lambda\int dx\, J_R^z J_L^z = -g\lambda\int dx\left(-\sin\theta_R M_R^x + \cos\theta_R M_R^z\right)\cdot$$

$$\cdot\left(\underbrace{-\sin\theta_L}_{-\sin\theta_R} M_L^x + \underbrace{\cos\theta_L}_{=\,-\cos\theta_R} M_L^z\right) =$$

$$= -g\lambda\int dx\left\{\sin^2\theta_R M_R^x M_L^x - \cos^2\theta_R M_R^z M_L^z + \cos\theta_R\sin\theta_R\left(M_R^x M_L^z - M_R^z M_L^x\right)\right\}$$

average over oscillations here

$$\Rightarrow (-g\lambda)\int dx\left\{\frac{1}{2}\sin^2\theta_R\left(\tilde{M}_R^x\tilde{M}_L^x - \tilde{M}_R^y\tilde{M}_L^y\right) - \cos^2\theta_R M_R^z M_L^z\right\}$$

$$= -g\lambda\int_x \left\{\frac{1}{4}\sin^2\theta_R\left(\tilde{M}_R^+\tilde{M}_L^+ + \tilde{M}_R^-\tilde{M}_L^-\right) - \cos^2\theta_R M_R^z M_L^z\right\}.$$

$$\Rightarrow \text{Full } H_{bs} = -g_{bs} \int_x \left(\cos\theta^- + \lambda \overset{\frac{1}{2}(1+\cos 2\theta_R) = \frac{1}{2}(1-\cos\theta^-)}{\cos^2\theta_R} \right) M_R^z M_L^z$$

$$+ \left[\frac{1}{4}(\cos\theta^- - 1) + \frac{1}{4}\lambda \underset{\frac{1}{2}(1-\cos 2\theta_R) = \frac{1}{2}(1+\cos\theta^-)}{\sin^2\theta_R} \right] (\tilde{M}_R^+ \tilde{M}_L^+ + h.c.)$$

$$\cos\theta^- + \lambda\, \frac{1}{2}(1-\cos\theta^-) = -\frac{1}{2}\lambda + \left(1+\frac{1}{2}\lambda\right)\cos\theta^- = \left(1+\frac{1}{2}\lambda\right)\cos\theta^- - \frac{\lambda}{2}.$$

$$\frac{1}{4}\left[\cos\theta^- - 1 + \lambda\,\frac{1}{2}(1+\cos\theta^-)\right] = \frac{1}{4}\left[\left(1+\frac{\lambda}{2}\right)\cos\theta^- - 1 + \frac{\lambda}{2}\right]$$

$$\pi v\, y_c = \frac{\pi v}{2}(y_x - y_y) = \frac{1}{2}\,\frac{-g_{bs}}{2\cancel{}}\left[\left(1+\frac{\lambda}{2}\right)\cos\theta^- + \frac{\lambda}{2} - 1\right] \qquad (GA)$$

Thus
$$H_{bs} = -g_{bs}\int dx \left\{ \left[\left(1+\frac{\lambda}{2}\right)\cos\theta^- - \frac{\lambda}{2}\right] M_R^z M_L^z \right.$$
$$\left. + \frac{1}{4}\left[\left(1+\frac{\lambda}{2}\right)\cos\theta^- - 1 + \frac{\lambda}{2}\right] \underbrace{\left(\tilde{M}_R^+ \tilde{M}_L^+ + \tilde{M}_R^- \tilde{M}_L^-\right)}_{2(M_R^x M_L^x - M_R^y M_L^y)} \right\}$$

$$\theta^- = 2\tan^{-1}\left(\frac{d}{h}\right)$$

$$H_{bs} = -\int dx \left(g_z M_R^z M_L^z + g_x M_R^x M_L^x + g_y M_R^y M_L^y \right)$$

Initially, $g_y(0) = -g_x(0)$.

$$\sum_a g_a \, M^a_{R\,Y} M^a_{L\,Y} \; \varepsilon_Y \varepsilon_{Y+1} = \sum_a g_a \; M^a_R \; \frac{-i N^a_Y}{4\pi(v\tau + ix)} \; \varepsilon_{Y+1} =$$

$$= \sum_a g_a \; \frac{(-i)}{4\pi(v\tau+ix)} \; \frac{(-i\,\varepsilon_Y)\,\varepsilon_{Y+1}}{4\pi(v\tau - ix)} = -\sum_a g_a \; \frac{\varepsilon_Y \varepsilon_{Y+1}}{(4\pi)^2 |z|^2}$$

$\Rightarrow \quad \dfrac{dG_\varepsilon}{dl} = G_\varepsilon \left(1 - \frac{1}{2}(y_x + y_y + y_z)\right)$ for the interchain $G_\varepsilon \, \varepsilon_Y \varepsilon_{Y+1}$ term.

$$\frac{dG_z}{dl} = G_z\left(1 + \frac{1}{2}(y_x + y_y - y_z)\right)$$

$$\frac{dG_x}{dl} = G_x\left(1 + \frac{1}{2}(y_y + y_z - y_x)\right)$$

$$\frac{dG_y}{dl} = G_y\left(1 + \frac{1}{2}(y_x + y_z - y_y)\right)$$

$$\frac{d}{dl} g_c = -\sum_{a,b \neq c} \frac{g_a g_b}{4\pi v}$$

$$\dot{g}_z = -\frac{g_x g_y}{2\pi v}, \quad \dot{g}_x = -\frac{g_y g_z}{2\pi v}, \quad \dot{g}_y = -\frac{g_x g_z}{2\pi v}$$

$$\Rightarrow \frac{d}{dl}(g_x + g_y) = -(g_x + g_y)\,\frac{g_z}{2\pi v}$$

$$\Rightarrow (g_x + g_y)_l = \underbrace{(g_x + g_y)_{l=0}}_{\overset{!!}{0}} \exp\left(-\int_0^l dx \, \frac{g_z(x)}{2\pi v}\right) = 0$$

$$\Rightarrow \underline{g_x(l) = -g_y(l)}$$

$$\Rightarrow \begin{cases} \dfrac{d}{dl} G_a = G_a\left(1 - \frac{1}{2} y_z\right) \quad \text{for } a = \varepsilon, z \qquad G_a = \dfrac{J'_a}{2\pi v} \\[2ex] \dfrac{d}{dl} G_x = G_x\left(1 + \dfrac{y_z - 2y_x}{2}\right), \quad \dfrac{d}{dl} G_y = G_y\left(1 + \dfrac{y_z + 2y_x}{2}\right). \end{cases}$$

$$\frac{d}{dl}(g_x - g_y) = \frac{(g_x - g_y)\, g_z}{2\pi v}$$

let $g_c = \frac{1}{2}(g_x - g_y)$, $y_c = \frac{g_c}{2\pi v}$

$$\Rightarrow \boxed{\dot{y}_c = y_c y_z \quad , \; \dot{y}_z = y_c^2}$$

$$\dot{g}_z = -\frac{g_x g_y}{2\pi v} = \frac{g_x^2}{2\pi v} \quad \text{but} \quad g_c = \frac{1}{2}\left(g_x - (-g_x)\right) = g_x$$

$$\Rightarrow \dot{y}_z = y_c^2$$

let $Y = y_c^2 - y_z^2$, $\dot{Y} = 2y_c(y_c y_z) - 2y_z(y_c^2) = 0$

$$\dot{y}_z = y_c^2 = Y + y_z^2$$

$$\int_{y_z(0)}^{y_z(l)} \frac{dy_z}{y_z^2 + Y} = \int_0^l dl = l$$

. Note $y_z^2 + Y = y_c^2 > 0$.

$$y_z(0) = \frac{g_{bs}}{2\pi v}\left[\left(1+\frac{\lambda}{2}\right)\cos\theta^- - \frac{\lambda}{2}\right] , \; y_c(0) = \frac{g_{bs}}{4\pi v}\left[\left(1+\frac{\lambda}{2}\right)\cos\theta^- - 1 + \frac{\lambda}{2}\right]$$ (GA420)

$$\cancel{y_c(0) = y_x(0) = \frac{1}{2}\left(\left(1+\frac{\lambda}{2}\right)\cos\theta^- - 1 + \frac{\lambda}{2}\right)}$$

$$\cancel{y_z(0) = \left(1+\frac{\lambda}{2}\right)\cos\theta^- - \frac{\lambda}{2}}$$

$$y_z(0) = \frac{g_{bs}}{2\pi v}\left\{\left(1+\frac{\lambda}{2}\right)\underbrace{\left(2\cos^2\left[\tan^{-1}\left(\frac{d}{h}\right)\right] - 1\right)}_{\frac{2h^2}{h^2+d^2} - 1 = \frac{h^2-d^2}{h^2+d^2}} - \frac{\lambda}{2}\right\} =$$

$$= \frac{g_{bs}}{2\pi v}\left\{\left(1+\frac{\lambda}{2}\right)\frac{h^2-d^2}{h^2+d^2} - \frac{\lambda}{2}\right\} = \frac{g_{bs}}{2\pi v}\left(\frac{h^2-d^2}{h^2+d^2} - \frac{\lambda d^2}{h^2+d^2}\right) \Rightarrow \frac{g_{bs}}{2\pi v}\,\frac{h^2 - (1+\lambda)d^2}{h^2+d^2}$$

$$y_c(0) = \frac{g b_s}{4\pi v}\left\{\left(1+\frac{\lambda}{2}\right)\frac{h^2-d^2}{h^2+d^2} - 1 + \frac{\lambda}{2}\right\} = \left(\eta \equiv \frac{g b_s}{2\pi v}\right) =$$

$$= \frac{\eta}{2}\left\{\frac{-2d^2}{h^2+d^2} + \frac{\lambda}{2}\,\frac{2h^2}{h^2+d^2}\right\} = \frac{\eta}{2}\,\frac{\lambda h^2 - 2d^2}{h^2+d^2} \Rightarrow y_c(0)<0$$

for most h.

$$Y = y_c^2 - y_z^2 = \left(\frac{\eta}{h^2+d^2}\right)^2\left[\left(\frac{\lambda}{2}h^2-d^2\right)^2 - \left(h^2-(1+\lambda)d^2\right)^2\right] =$$

$$= \frac{\eta^2}{(h^2+d^2)^2}\left[\frac{\lambda^2 h^4}{4} + d^4 - \lambda h^2 d^2 - \right.$$

$$\left. - \left(h^4 + (1+\lambda)^2 d^4 - 2(1+\lambda)h^2 d^2\right)\right] =$$

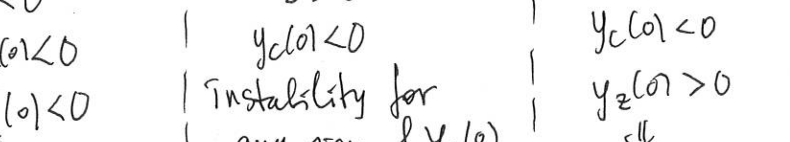
$$= \left(\frac{\eta}{h^2+d^2}\right)^2\left[\left(1-(1+\lambda)^2\right)d^4 + (2+\lambda)h^2d^2 - \left(1-\frac{\lambda^2}{4}\right)h^4\right]$$

$$= \left(\frac{\eta}{h^2+d^2}\right)^2\left[2h^2d^2 - h^4 + \lambda\left(h^2d^2 - 2d^4\right)\right] + O(\lambda^2) =$$

$$h^2\left(2d^2-h^2\right) + \lambda d^2\left(h^2-2d^2\right)$$

$$= \left(\frac{\eta}{h^2+d^2}\right)^2\left(h^2-\lambda d^2\right)\left(-h^2+2d^2\right)$$ to linear in $\lambda \sim d^2$ order.

$$= \frac{\eta^2\left(h^2-\lambda d^2\right)\left(2d^2-h^2\right)}{(h^2+d^2)^2}$$



$Y<0$ $y_c(0)<0$ $y_z(0)<0$ No instability	$Y>0$ $y_c(0)<0$ Instability for any sign of $Y_z(0)$ But $y_z(\infty)>0$ $y_c(\infty)<0$	$Y<0$ $y_c(0)<0$ $y_z(0)>0$ $\Downarrow$ Instability $y_c(\infty)<0$	Make plots of Y, $y_z(0)$, $y_c(0)$ as funct. of h/d

(1) Suppose $Y > 0$

$$\int \frac{dy_z}{\left(y_z + i\sqrt{Y}\right)\left(y_z - i\sqrt{Y}\right)} = \int dy_z \left(\frac{1}{y_z - i\sqrt{Y}} - \frac{1}{y_z + i\sqrt{Y}} \right) \frac{1}{2i\sqrt{Y}}$$

$$= \frac{1}{2i\sqrt{Y}} \ln \frac{y_z(l) - i\sqrt{Y}}{y_z(l) + i\sqrt{Y}} \cdot \frac{y_z(0) + i\sqrt{Y}}{y_z(0) - i\sqrt{Y}} = l$$

$$\frac{y_z(l) - i\sqrt{Y}}{y_z(l) + i\sqrt{Y}} = e^{i2\sqrt{Y}l} \frac{y_z(0) - i\sqrt{Y}}{y_z(0) + i\sqrt{Y}}$$

$$y_z - i\sqrt{Y} = e\left(y_z + i\sqrt{Y}\right), \quad e \equiv \frac{y_z(0) - i\sqrt{Y}}{y_z(0) + i\sqrt{Y}} e^{i2\sqrt{Y}l}$$

$$y_z(1 - e) = (1 + e) i\sqrt{Y}$$

$$y_z(l) = \frac{1 + e^{i2\sqrt{Y}l}}{1 - e^{i2\sqrt{Y}l}} i\sqrt{Y} = \frac{2\cos\sqrt{Y}l}{-2i\sin\sqrt{Y}l} i\sqrt{Y}$$

$$y_z(l) = -\sqrt{Y}\,\mathrm{cotan}\left(\sqrt{Y}l\right).$$

$$y_c^2(l) = Y + y_z^2 = Y + Y\frac{\cos^2}{\sin^2} = \frac{Y}{\sin^2\left(\sqrt{Y}l\right)}.$$

$$y_z(l) = i\sqrt{Y} \frac{y_z(0) + i\sqrt{Y} + \left(y_z(0) - i\sqrt{Y}\right)e^{i2\sqrt{Y}l}}{y_z(0) + i\sqrt{Y} - \left(y_z(0) - i\sqrt{Y}\right)e^{i2\sqrt{Y}l}} =$$

$$= i\sqrt{Y} \frac{\left(y_z(0) + i\sqrt{Y}\right)e^{-i\sqrt{Y}l} + \left(y_z(0) - i\sqrt{Y}\right)e^{i\sqrt{Y}l}}{\left(y_z(0) + i\sqrt{Y}\right)e^{-i\sqrt{Y}l} - \left(y_z(0) - i\sqrt{Y}\right)e^{i\sqrt{Y}l}} =$$

$$= i\sqrt{Y} \frac{2\cos\sqrt{Y}l\, y_z(0) + i\sqrt{Y}\left(-2i\sin\sqrt{Y}l\right)}{-2i\sin\sqrt{Y}l\, y_z(0) + i\sqrt{Y}\, 2\cos\sqrt{Y}l}$$

$$\Rightarrow \quad y_z(\ell) = \sqrt{Y}\ \frac{\cos\sqrt{Y}\ell\ y_z(0) + \sqrt{Y}\sin\sqrt{Y}\ell}{\cos\sqrt{Y}\ell\ \sqrt{Y} - \sin\sqrt{Y}\ell\ y_z(0)}$$

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So that $y_z(0) = y_z(0)$ of course.

denominator $=0$ when $\tan\sqrt{Y}\ell_0 = \dfrac{\sqrt{Y}}{y_z(0)} = \sqrt{\left(\dfrac{y_e(0)}{y_z(0)}\right)^2 - 1}$

$$y_z(\ell) = \sqrt{Y}\ \frac{y_z(0) + \sqrt{Y}\tan\sqrt{Y}\ell}{\sqrt{Y} - y_z(0)\tan\sqrt{Y}\ell} \xrightarrow[\ell\to\ell_0]{} \sqrt{Y}\ \frac{y_z(0) + \frac{Y}{y_z(0)}}{\sqrt{Y}\left[1 - \frac{y_z(0)}{\sqrt{Y}}\tan\sqrt{Y}\ell\right]} =$$

$$\cancel{y_e^2(\ell) = Y^2 + y_z^2(\ell) = Y\left\{\left(\frac{\cos y_z - \sin\sqrt{Y}}{\cos\sqrt{Y} - \sin y_z}\right)^2 + 1\right\} = Y}$$

$$= \frac{y_e^2(0)}{y_z(0)}\ \frac{1}{1 - \frac{y_z(0)}{\sqrt{Y}}\tan\sqrt{Y}\ell} > 0$$

$$\cancel{\Rightarrow \sqrt{Y}\ell_0 = \frac{\pi}{2},\ \ell_0 = \frac{\pi}{2\sqrt{Y}}}$$

$$\Rightarrow \quad \sqrt{Y}\ell_0 = \tan^{-1}\frac{\sqrt{Y}}{y_z(0)}.$$

For $Y \to 0$ $\quad \ell_0 \to \dfrac{1}{y_z(0)}$, $\quad y_z(\ell) \to \dfrac{y_z(0) + \cancel{Y\ell}}{1 - \ell\, y_z(0)}$.

provided $y_z(0) > 0$ of course.

(2) Suppose $Y < 0$, $Y = -\mu^2$, $\mu = \sqrt{y_z^2(0) - y_c^2(0)}$

$$\int \frac{dy_z}{y_z^2 - \mu^2} = \int \frac{dy_z}{(y_z+\mu)(y_z-\mu)} = \int dy_z \left(\frac{1}{y_z-\mu} - \frac{1}{y_z+\mu} \right) \frac{1}{2\mu}$$

$$= \frac{1}{2\mu} \ln \frac{y_z(l)-\mu}{y_z(l)+\mu} \frac{y_z(0)+\mu}{y_z(0)-\mu} = l$$

$$\frac{y_z(l)-\mu}{y_z(l)+\mu} = \frac{y_z(0)-\mu}{y_z(0)+\mu} e^{2\mu l}$$

$$y - \mu = e(y+\mu), \quad y(1-e) = (1+e)\mu$$

$$y_z(l) = \mu \frac{y_z(0)+\mu + (y_z(0)-\mu)e^{2\mu l}}{y_z(0)+\mu - (y_z(0)-\mu)e^{2\mu l}}$$

$$y_z(l) = \mu \frac{(y_z(0)+\mu)e^{-\mu l} + (y_z(0)-\mu)e^{\mu l}}{(y_z(0)+\mu)e^{-\mu l} - (y_z(0)-\mu)e^{\mu l}} =$$

$$= \mu \frac{ch(\mu l)\, y_z(0) - \mu\, sh(\mu l)}{-sh(\mu l)\, y_z(0) + \mu\, ch(\mu l)}$$

$$\frac{sh(\mu l)}{ch(\mu l)} = \frac{\mu}{y_z(0)} = \frac{\sqrt{y_z^2 - y_c^2}}{y_z}$$ $\Rightarrow$ divergence is ~~poss~~ present if $y_z(0) > 0$.

$$sh(\mu l) = \sqrt{ch^2(\mu l) - 1}$$

$$\frac{ch^2 - 1}{ch^2} = \frac{y_z^2 - y_c^2}{y_z^2} \Rightarrow ch^2(\mu l) = \frac{y_z^2}{y_c^2}, \quad ch(\mu l) = \frac{y_z}{y_c}.$$

$$l_0 = \frac{1}{\sqrt{y_z^2 - y_c^2}} \cosh^{-1}\left(\frac{y_z}{y_c}\right)$$

Since $g_x(l) = -g_y(l)$, $\quad \dot{G}_{\varepsilon,z} = G_{\varepsilon,z}\left(1 - \frac{1}{2} y_z\right)$.

$$\dot{G}_{x} = G_{x}\left(1 + \frac{1}{2}(y_z - 2y_c)\right)$$

$$\dot{G}_y = G_y\left(1 + \frac{1}{2}(y_z + 2y_c)\right)$$

with $y_c(l\to\infty) < 0$ $\quad G_x(l) > G_y(l)$

(and as long as it flows to strong coupling) $\quad G_x(l) > G_{\varepsilon,z}(l)$

But if $y_c(\infty) = 0$ (weak coupling) $\Rightarrow y_z(\infty) < 0 \Rightarrow G_{\varepsilon,z} > G_x$?!

this is really $l_d = \ln \frac{v}{D}$

$$G_x(l) = G_x(0)\exp\left[l + \int_0^l dl\ \frac{1}{2}(y_z - 2y_c)\right]$$

$$G_y(l) = G_y(0)\exp\left[l + \int dl\ \frac{1}{2}(y_z + 2y_c)\right]$$

$$G_{\varepsilon,z}(l) = G_{\varepsilon,z}(0)\exp\left[l - \int_0^l dl\ \frac{1}{2} y_z\right]$$

3 phases

$Y > 0$

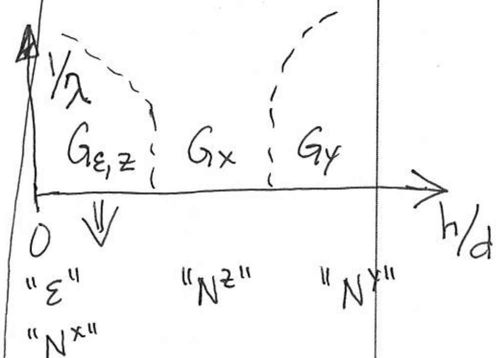
$y_c(0) < 0 \to y_c(l) < 0$
$y_z(0) > 0 \to y_z(l) > 0$ $\Rightarrow G_x(l)$ wins

$Y > 0$

$y_c(0) < 0 \to y_c(l) \not< 0$
$y_z(0) < 0 \to y_z(l) > 0$ $\Rightarrow G_x(l)$

$\lambda = \frac{c}{2}\frac{D^2}{J^2}$, $c = \left(\frac{2}{\pi}\frac{2\pi v}{g_{bs}}\right)^2$

Initial value: $\frac{g_{bs}}{2\pi v} \approx 0.23$



$Y < 0$ this is LL regime in single chain.

$y_c(0) < 0 \to y_c(l) \to 0$
$y_z(0) < 0 \to y_z(l) < 0$ $\Rightarrow G_{\varepsilon,z}(l)$

$Y < 0$

$y_c(0) > 0 \to y_c(l) > 0$
$y_z(0) > 0 \to y_z(l) > 0$ $\Rightarrow G_y(l)$

$h \perp D$ DM scale $q_0^{-1} = \frac{v}{\sqrt{h^2+d^2}} = e^{l_d}$.

(1) For $0 < l < l_d$ the RG flow is not affected by oscillatrons $\Rightarrow$ $g_a(l) = \dfrac{g_a(0)}{1 + l\,\frac{g_a(0)}{2\pi v}}$

$$\Rightarrow \quad y_a(l_d) = \frac{y_a(0)}{1 + l_d\, y_a(0)} \quad , \; y_a = \frac{g_a}{2\pi v}$$

(2) For $l > l_d$ $y_a(l_d)$ serves as initial value.

e^{iq_0x} oscillatrons are important and modify H_{bs} and H_{inter} as discussed on previous pages.

H_{bs} flows now to SDW order, but inter-chain interactions interrupt this.

Here $\underbrace{y_z(0) = y_z(l_d)}_{\text{actually}} = \eta_d \dfrac{h^2 - (1+\lambda)d^2}{h^2+d^2}$

$$\eta_d \equiv \eta(l_d) = \frac{\eta}{1 + l_d\,\eta} \quad , \; \eta = \frac{g_{bs}}{2\pi v} = 0.23$$

$$\lambda = \frac{1}{2}\left(\frac{2}{\pi\eta}\right)^2 \frac{d^2}{J^2} = \frac{c}{2} d^2 . \quad (\text{set } J=1)$$

$$y_c(0) = y_c(l_d) = \frac{1}{2}\eta_d \frac{\lambda h^2 - 2d^2}{h^2+d^2}$$

$$Y = y_c^2 - y_z^2 = \eta_d^2 \frac{(h^2 - \lambda d^2)(2d^2 - h^2)}{(h^2+d^2)^2}$$

$$\cancel{\frac{y}{1+ly} - \frac{1}{l} = \frac{yl - (1+ly)}{l(1+ly)} = -\frac{1}{l(1+ly)}}.$$

$$\frac{G_{\varepsilon,z}(l)}{G_x(l)} = \frac{G_{\varepsilon,z}(0)}{G_x(0)} \frac{e^{l - \int_0^l dt \, \frac{1}{2} y_z(t)}}{e^{l + \frac{1}{2}\int_0^l dt (y_z - 2y_c)}} =$$

$$= \frac{J' \frac{v}{\sqrt{h^2+d^2}} \; \frac{1}{2} \sin^2\theta_R}{J' \frac{v}{\sqrt{h^2+d^2}}} \exp\left\{\int_0^l dt \left(y_c(t) - y_z(t)\right)\right\}.$$

$$\frac{G_y(l)}{G_x(l)} = \sin^2\theta_R \; \exp\left\{\int_0^l dt \; 2y_c(t)\right\}.$$

$$\sin^2\theta_R = \frac{h^2}{h^2+d^2}.$$

essentially we need $y_c(t) > 0$ which is possible for

$$\lambda h^2 > 2d^2 \quad , \; h^2 > \frac{2d^2}{\lambda} = \frac{4}{c} \quad , \; h > \frac{2}{\sqrt{c}} = \frac{2}{\frac{2}{\pi\eta}} = \pi\eta = \frac{g_{bs}}{2v}.$$

let $Y<0$ and $y_z(0)<0$ $\left(Y=-\mu^2,\ y_z(0)=-|y_z(0)|\right)$

$$y_z(\ell) = -\mu \frac{|y_z(0)|\, ch(\mu\ell) + \mu\, sh(\mu\ell)}{|y_z(0)|\, sh(\mu\ell) + \mu\, ch(\mu\ell)} \xrightarrow[\ell\to\infty]{} -\mu$$

then $y_c(\ell) \to 0$ in the same limit.

$$\frac{G_{\xi,z}(\ell)}{G_x(\ell)} \approx \frac{h^2}{2(h^2+d^2)} \exp(\mu\ell)$$

$$\mu^2 = \eta_d^2 \frac{(\lambda d^2 - h^2)(2d^2 - h^2)}{(h^2+d^2)^2} \underset{h\sim d^2}{\approx} 2\eta_d^2\left(\lambda - \frac{h^2}{d^2}\right).$$

$\Downarrow$ critical

$h_c^2 = \lambda d^2 = \frac{1}{2} c d^4.$

$$\Rightarrow \frac{h^2}{2d^2} \exp\left(\eta_d \sqrt{2\lambda}\, \ell_2\right) = 1$$

$$\ell_2 = \frac{1}{\eta_d\sqrt{2\lambda}} \ln\left(\frac{2d^2}{h^2}\right) = \frac{1}{\eta_d\left(\frac{2}{\pi\eta}\right) d} \ln\left(\frac{4d^2}{cd^4}\right)$$

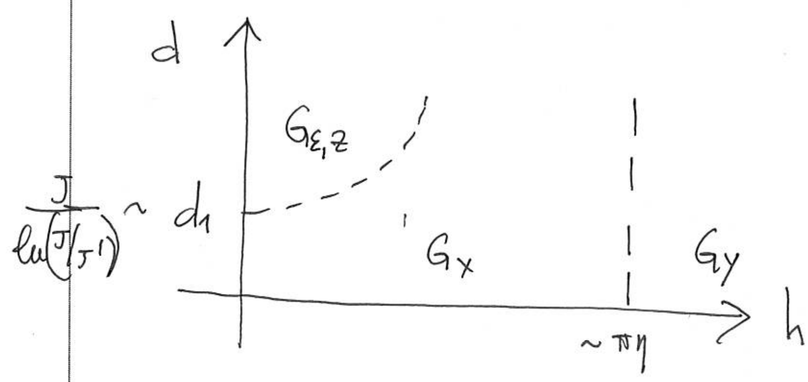
$$\Rightarrow \ell_2 \approx \frac{\pi J}{2D} \ln\left(\frac{\pi^2\eta^2}{d^2}\right) . \underbrace{\left[\times \ln\frac{2\pi v}{d}\right]}_{\text{from } \frac{1}{\eta_d}\ ?}$$

Strong coupling due to J' takes place at $\ell' = \ln\left(\frac{2\pi V}{J'}\right) \sim \ln\left(\frac{J}{J'}\right) \Rightarrow$ we need $\ell_2 < \ell'$

for $G_{\varepsilon,z}(\ell)$ to overtake $G_x(\ell)$ on this scale

$\Rightarrow \frac{J}{D} < \ln\left(\frac{J}{J'}\right) \Rightarrow D > \frac{J}{\ln(J/J')}$. For $D <$ that critical value G_x wins!

$d \perp h$



Since $\pi\eta \sim O(1)$, the $G_x - G_y$ transition is tentative.
Can probably be studied from the large-h (saturation) limit?

<u>G_x phase</u> : $G_x \hat{N}^x_y \hat{N}^x_{y+1}$ is the leading term $\Rightarrow \langle \hat{N}^x_y \rangle = (-1)^y \tilde{N}_1$

<u>p. 19:</u> $\hat{N}^x_y = N^z_y \Rightarrow$ this corresponds to "N^z" order in the original basis.

$$\int_x G_x \tilde{N}^x_y \tilde{N}^x_{y+1} = G_x \int_x \left(-A_3 \sin\sqrt{2\pi}\,\hat{\theta}_y\right)\left(-A_3 \sin\sqrt{2\pi}\,\hat{\theta}_{y+1}\right)$$

it is not affected by the shift of $\hat{\varphi}$ fields
$\Rightarrow$ describes <u>commensurate</u> order

$$\langle \vec{S}_y(x) \rangle = e^{i\pi x} \langle \hat{N}^x_y \rangle = (-1)^x (-1)^y \tilde{N}_1$$

<u>G_y phase:</u> $\langle \hat{N}^y_y \rangle = (-1)^y \tilde{N}_2$

$\Rightarrow$ this is state with "N^y" $\neq 0$ and "ε" $\neq 0$ in the original frame.

p. 20 $$\langle \vec{S}_y(x) \rangle = e^{i\pi x} \left(-\sin\theta_R \langle \hat{N}^y_y \rangle\right) = -\sin\theta_R (-1)^{x+y} \tilde{N}_2$$

$G_x - G_y$ is a spin-flop transition. G_y dimerized GS (dimerization is staggered too) like $(-1)^y$