Lorentz Transformations

This handout summarizes the lecture discussion of Lorentz transformations.

1 Introduction

When two observers are moving relative to each other, we use Lorentz transformations to relate their observations.

2 Four vectors

Lorentz transformations operate on **four-vectors** (more generally tensors with four-vector indices). An example of a four-vector is the **space-time coordinate**, formed from the time coordinate t and the three-dimensional vector space coordinate $\mathbf{x}$. We multiply the time by the velocity of light.

$$x = \begin{pmatrix} ct \\ \mathbf{x} \end{pmatrix}$$

We can label the components of the four-vector as x^μ with μ ranging from 0 to 3. The superscript is conventional. Then the time component is $x^0 = ct$. Another common four-vector is the **four-momentum**

$$p = \begin{pmatrix} E \\ \mathbf{p}c \end{pmatrix}$$

The component p^0 is the energy (which must include the rest energy). The four-momentum of a particle of mass m at rest is just

$$p = \begin{pmatrix} mc^2 \\ \mathbf{0} \end{pmatrix}$$

3 Lorentz transformation

The Lorentz transformation is the central feature of special relativity that was adopted in order to account for the remarkable observation that the speed of a light ray measured by an observer was apparently the same, regardless of how fast the observer was moving.

Suppose observers in frames S and S' are moving with a velocity v relative to each other. To be more precise, let's align the spatial coordinate systems for each observer, so S' is moving along the z-axis of observer S and the x and y axes are parallel with the x' and y' axes. Suppose, also that both observers set their clocks so that when their origins are on top of each other, their clocks read 0 also.

Then if an observer in frame S sees an event at the space time coordinate x^μ, and an observer S' sees the same event at space-time coordinate $x^{\mu\prime}$, the coordinates are related by the Lorentz transformation

$$\begin{aligned} x^{0\prime} &= \gamma(x^0 - \beta x^3) \\ x^{1\prime} &= x^1 \\ x^{2\prime} &= x^2 \\ x^{3\prime} &= \gamma(x^3 - \beta x^0). \end{aligned}$$

We have introduced

$$\begin{aligned} \beta &= v/c \\ \gamma &= \frac{1}{\sqrt{1-\beta^2}}. \end{aligned}$$

In terms of t, x, y, and z, the Lorentz transformation is

$$\begin{aligned} ct' &= \gamma(ct - \beta z) \\ x' &= x \\ y' &= y \\ z' &= \gamma(z - \beta ct) \end{aligned}$$

As we mentioned, this transformation assures that if both observers see the same light ray and measure its speed, they both get the same result, namely c. A simple way to check this is to suppose that the light ray leaves the origin at time $t = 0$. Later on, the observer S notices that it has reached the space-time coordinate x^μ, while observer S' notices that it has reached x'^μ. The relationship between these observations is given by the Lorentz transformation. It is easy to check the **Lorentz invariance** property

$$\begin{aligned} (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 &= (x^{0\prime})^2 - (x^{1\prime})^2 - (x^{2\prime})^2 - (x^{3\prime})^2 \quad \text{or'} \\ (ct)^2 - r^2 = (ct')^2 - (r')^2 \end{aligned}$$

where we have introduced the distance to the origin $r = \sqrt{x^2 + y^2 + z^2}$ for observer S and $r' = \sqrt{(x')^2 + (y')^2 + (z')^2}$ for observer S'. For a light ray moving with velocity c we must have $r = ct$. So both sides of the equation above must be zero. More gnerally, Both observers always get the same value for any Lorentz invariant quantity! So we conclude that also $r' = ct'$, and both observers agree that the speed is c.

If the relative speed v is much less than the speed of light, we have $\gamma \approx 1$ and

$$\begin{aligned} ct' &= ct \\ x' &= x \\ y' &= y \\ z' &= z - vt \end{aligned}$$

which is the Galilean relativity that we learned in our first course in classical mechanics.

4 Time dilation and Lorentz-Fitzerald contraction

As an application of a space-time Lorentz transformation, consider how time intervals are recorded by different observers. As a specific application, suppose a particle is placed at rest at the origin of observer S at time $t = t' = 0$. It decays after a period of time τ. Observer S' also sees the decay. According to his/her clocks, what is the elapsed time?

The answer is easily found from the relation $ct' = \gamma(ct - \beta z)$, which we apply to the decay event. Notice that to use this relation, we have to specify z. Since the decaying particle is still at the S origin, we have $z = 0$, and $t' = \gamma\tau$. So fast moving particles appear to have longer lifetimes, according to the Lorentz factor γ. This is the peculiar phenomenon of **time dilation**.

As a second application, consider how distance measures are recorded by different observers. Suppose that observer S places a meter stick along the z axis and observer S' measures its length. What will the result be?

As always, we answer by considering the relationship between a pair of events. In this case the events are S' measuring the two ends of the meter stick, which she must do at the same instant t'. (Otherwise, she would have to correct for the motion of the meter stick.) We can use the Lorentz transformation equations given above without modification, if we suppose that S' makes the measurement at $t' = 0$ when the S and S' origins coincide. Both observers agree that the left end of the meter stick is at $z = z' = 0$ when $t = t' = 0$. S would say the right end of the meter stick is always at $z = 1$ and S' records it as z'. What seems peculiar is that they will disagree about the simultaneity of the events of measuring both ends. If we look at the relation $ct' = \gamma(ct - \beta z)$ we see that $t' = 0$ requires $ct = \beta z$, so only for an event at $z = 0$ or for for vanishingly low velocities $v/c \to 0$ do both observers agree on the time. So when observer S' measures the right end of the meter stick at time $t' = 0$, observer S says that measurement happens at time $ct = \beta z$ (with $z = 1$ meter.) But from the fourth relation $z' = \gamma(z - \beta ct)$, observer S' must be doing this measurement at position $z' = \gamma(z - \beta^2 z) = z/\gamma$. So to S' the meter stick is short by a factor $1/\gamma$. This is the **Lorentz-Fitzgerald contraction**.

Note that both observers always agree on lengths perpendicular to their relative velocity. The contraction is only along the direction of relative motion.

5 Lorentz transformations and other four-vectors

The four-momentum transforms just like the space-time coordinate under a Lorentz transformation:

$$\begin{aligned}
p^{0\prime} &= \gamma(p^0 - \beta p^3) \\
p^{1\prime} &= p^1 \\
p^{2\prime} &= p^2 \\
p^{3\prime} &= \gamma(p^3 - \beta p^0).
\end{aligned}$$

so

$$\begin{aligned}
E' &= \gamma(E - \beta p_z c) \\
p'_x c &= p_x c \\
p'_y c &= p_y c \\
p'_z c &= \gamma(p_z c - \beta E)
\end{aligned}$$

In fact a four-vector is defined to be any quantity that transforms in this way.

6 Velocity transformation

Let's see how velocities transform under a Lorentz transformation. We consider a particle moving along the z axis with $z = wt$ according to observer S. He says it has velocity w. Observer S', moving with velocity v relative to S, says it has velocity $w' = z'/t'$. We want to find it. Now we are dealing with three velocities – the velocities of the particle as seen by the observers and the relative velocity of the observers. For simplicity let's do this in only one dimension, so suppose all velocities are along the z axis. Then a Lorentz transformation relates the position of the particle as seen by both observers:

$$\begin{aligned} z' &= \gamma(z - \beta ct) = \gamma(w - v)t \\ ct' &= \gamma(ct - \beta z) = \gamma(c - vw/c)t \quad . \end{aligned}$$

We divide the equations to get

$$w' = \frac{w - v}{1 - vw/c^2} \quad .$$

In the low velocity limit we recover the familiar Galilean result $w' = w - v$. But at the other extreme, if $w = c$, we get $w' = c$, so all observers agree on the same velocity for a particle moving at the speed of light.

7 Lorentz invariants

We have already seen that the quantity

$$x^2 = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 = (ct)^2 - \mathbf{x}^2$$

is invariant under a Lorentz transformation. By the same token the quantity

$$p^2 = (p^0)^2 - (p^1)^2 - (p^2)^2 - (p^3)^2 = E^2 - \mathbf{p}^2c^2 = m^2c^4$$

is also invariant, since it transforms the same way. This result is fortunate, since then all observers agree on the same mass. This relativistic relationship between energy, momentum, and mass is sometimes called the **mass shell condition**.

More general invariants can be constructed from any pair of four-vectors. For example, if we have two four-momenta k and p, the combination

$$p \cdot k = p^0k^0 - p^1k^1 - p^2k^2 - p^3k^3$$

is also invariant. Notice that this expression defines an inner product rather like a dot product, but the signs (metric) are different from the familiar Cartesian dot product. The familiar dot product of three-dimensional vectors is similarly invariant under rotations.

8 Photon four-momentum

A photon has zero mass. If it has frequency ν, then its wavelength is $\lambda = c/\nu$ and (if it is moving in the z direction), its energy and momentum are given by

$$p^\mu = \begin{pmatrix} E \\ 0 \\ 0 \\ pc \end{pmatrix}$$

where $E = h\nu$ and $p = h/\lambda = E/c$. This result is consistent with the statement that

$$p^2 = E^2 - (cp)^2 = 0,$$

meaning the photon has zero mass.

9 Relativistic kinematics

9.1 Kinetic Energy

A moving particle has kinetic energy. In Newtonian mechanics we thought of kinetic energy as being the primary quantity and then tacked on the rest mass when we began talking about low-velocity nuclear physics. In special relativity, we treat the total energy as the primary quantity, given by $E = \sqrt{m^2c^4 + \mathbf{p}^2c^2}$, and the kinetic energy K is then derived from it using

$$K = E - mc^2.$$

It is a useful exercise to check that the relativistic mass-energy-momentum relation above turns into the standard nonrelativistic result for speeds much less than the speed of light:

$$K = \sqrt{m^2c^4 + \mathbf{p}^2c^2} - mc^2 \approx \mathbf{p}^2/2m$$

9.2 Two-body collisions

Consider a two-body reaction of the type

$$1 + 2 \rightarrow 3 + 4. \tag{1}$$

Let the four-momenta of the particles be p_1, p_2, p_3, and p_4, respectively. Notice, we are now using subscripts to label the particle. Each of these terms represents a full four-vector. Let the masses similarly be m_1, m_2, m_3, and m_4. Energy and momentum must be conserved. The conservation condition can be written concisely in terms of four-vectors as

$$p_1 + p_2 = p_3 + p_4 \tag{2}$$

Suppose that particle 2 is at rest and particle 1 arrives along the positive z axis. Then the initial four-momenta are

$$p_1 = \begin{pmatrix} E \\ 0 \\ 0 \\ pc \end{pmatrix} \qquad p_2 = \begin{pmatrix} m_2c^2 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Of course we must also have $E^2 - p^2c^2 = m_1^2c^4$, so there is really only one kinematic parameter for the incoming particle. Let the scattering plane be the xz plane. Then the final four-momenta are

$$p_3 = \begin{pmatrix} E_3 \\ p_3c\sin\theta \\ 0 \\ p_3c\cos\theta \end{pmatrix} \qquad p_4 = \begin{pmatrix} E_4 \\ -p_3c\sin\theta \\ 0 \\ (p - p_3\cos\theta)c \end{pmatrix}$$

where we have already arranged for three-momentum conservation. The energies must also be related through the energy conservation condition:

$$E + m_2c^2 = E_3 + E_4.$$

This constraint together with the mass shell condition for particles 3 and 4 determines the outcome completely in terms of the one free parameter, namely, the scattering angle θ.

9.3 Compton Scattering

Let's suppose that we are considering Compton scattering, so that particles 1 and 3 are light particles and particles 2 and 4 are an electron. A light particle has zero mass, so we have $E = pc$ and $E_3 = p_3 c$. The mass shell condition for particle 4 then becomes

$$E_4^2 - (E_3 \sin\theta)^2 - (E - E_3 \cos\theta)^2 = m^2 c^4$$

which, together with energy conservation and a little algegra, gives for the energy of the scattered photon,

$$E_3 = \frac{Emc^2}{E(1-\cos\theta) + mc^2}.$$

9.4 Relating lab and c.m. frames

It is often helpful to analyze collisions in the c.m. frame. We can use Lorentz invariants to help relate quantities in the two frames. For example, consider the two-body reaction with any mass. In the c.m. frame let's write four-momenta of particles 1 and 2 this way:

$$p_1 = \begin{pmatrix} E_{1c} \\ 0 \\ 0 \\ p_c c \end{pmatrix} \qquad p_2 = \begin{pmatrix} E_{2c} \\ 0 \\ 0 \\ -p_c c \end{pmatrix}$$

The total energy in the c.m. frame is

$$W = E_{1c} + E_{2c}$$

The total four-momentum in the c.m. frame is then

$$P = p_1 + p_2 = \begin{pmatrix} W \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

We can use a trick with the Lorentz invariant P^2 to relate the total energy W in the c.m. to the energy E in the laboratory frame. We have

$$W^2 = P^2 = (p_1 + p_2)^2 = p_1^2 + 2p_1 \cdot p_2 + p_2^2$$

Since $p_1 \cdot p_2$ is a Lorentz invariant, we can evaluate it in the laboratory frame, where it is just Em_2c^2. So we get

$$W^2 = m_1^2c^4 + m_2^2c^4 + 2Em_2c^2$$

To get the momentum of each particle in the c.m. requires a bit more algebra. We just quote the result:

$$p_cc = \sqrt{W^4 + m_1^4c^8 + m_2^4c^8 - 2W^2m_1^2c^4 - 2W^2m_2^2c^4 - 2m_1^2m_2^2c^8}/(2W)$$

From the momentum we can then get the energies E_{1c} and E_{2c} for each particle.