

Name:

Section:

Part I: Types of Error- Accuracy vs. Precision

Through this and all other science labs you will take, you will find error as an inescapable feature of all measurements and calculations that you will do. The two types of error we will study today are first, random error, and second, systematic error.

Random errors are those errors that *can be revealed statistically in the analysis of repeated measurements*. Random error is a fundamental feature of measurements that arises from the physics of the process being measured and the device used for the measurement, even when the utmost care is taken during the measurement. Because such error is random, we expect equal probability that a given measurement fall above the "true" value as below it.

Systematic error arises when *something in the measuring process affects all measurements in an equal or consistent way* (examples include improper instrument calibration, alignment issues, poor electrical contact, etc.). Systematic errors are avoidable down to the level of the random errors in that the experimenter can remove such errors by more vigilant attention to instrument calibration, parallax, proper electrical contacts, etc.

To simplify these concepts, we can use accuracy and their precision, which we will define by the shot patterns of a rifle target. First, **accuracy** is *related to systematic error result, and tells you how close you are to the "true" values, or the values you are looking for*. In the figure below, we see the shots on the left target are not located near the target center. This is a low accuracy. The target on the right has a midpoint value very close to the target center, so we would say it has a high accuracy. Now, as you have already noticed, there is a difference between the scatter of both graphs. This scatter is related to the **precision**, which is *related to random error, and describes how repeatable an experiment is based on how much scatter it produces*. In our target below, we see that the left system has a low amount of scatter because all the points are located very close together, which means it has high precision. The target on the left has points all over the page, which would indicate it has low precision.

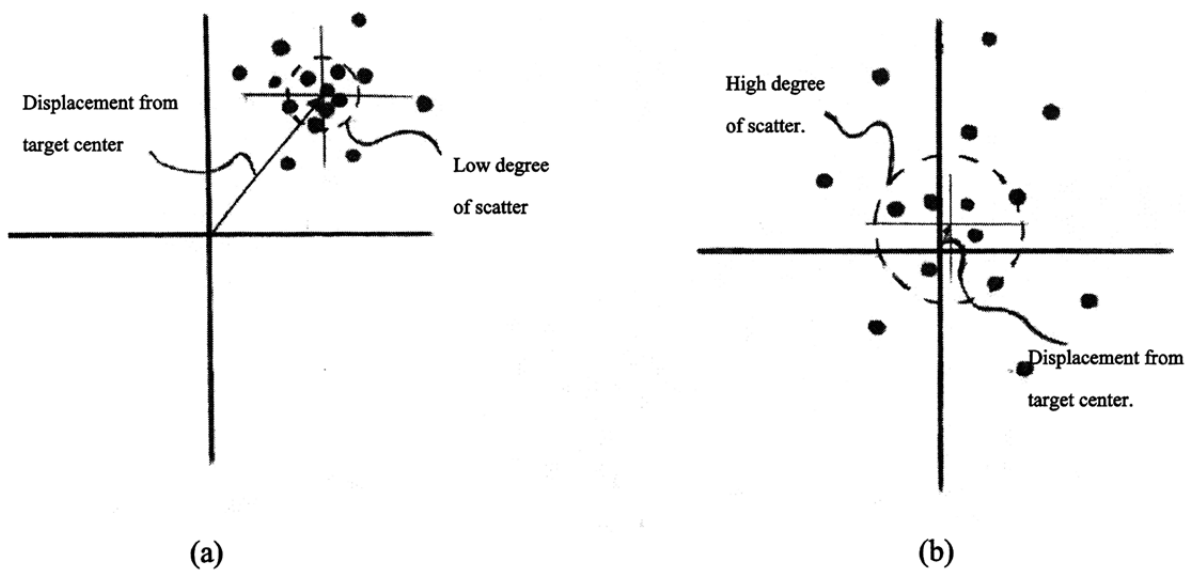


Figure 1: Accuracy vs. Precision

There are two other terms that we will use through this lab manual that are directly

related with accuracy and precision. We define **dispersion** or **scatter** to be *how far apart individual data points are located from one another*, which would make it closely related to *precision*. In fact, we can see that the two are inversely related, or that *high precision means low scatter, and vice-versa*. We will also use **deviation** to be *the value by which our calculated values differ from the actual or "true" value*, which makes it closely related to *accuracy*. In fact, these two are also inversely related, meaning *high accuracy results in low deviation*. You will need to be very careful to keep these terms straight and not mix them up in your mind!

Part II: Finding Uncertainty from Repeated Measurements

The best way to determine random error is to measure the same quantity many times, independently. Fluctuations of the various measurements about the mean or average give the uncertainty. For example, here are ten consecutive measurements of a length. Just looking at the spread of these data, we can guess that the uncertainty in a given measurement is about ± 0.1 cm, because most of them lie between 92.3 and 92.5.

92.3 cm	92.4 cm
92.4 cm	92.4 cm
92.3cm	92.4 cm
92.5 cm	92.6 cm
92.3 cm	92.5 cm

Table 1: Sample Length Data

Rather than guess the uncertainty by inspection, we can compute the uncertainty straight from the data. First we calculate the mean of the set of N measurements:

$$\bar{x} = \frac{x_1 + x_2 + \cdots + x_N}{N}$$

Then we go back and calculate the average deviation of the points from the mean. The deviation of a point from the mean is given by:

$$x_i - \bar{x}$$

However, we cannot just take the average of the deviations because the positive deviations will cancel the negative deviations. We must average the squared deviations to eliminate the negative signs, then take the square root:

$$\delta x = \sqrt{\frac{(x_1 - \bar{x})^2 + (x_2 - \bar{x})^2 + \cdots + (x_N - \bar{x})^2}{N - 1}}$$

This is called the mean-squared deviation and is the measure of the expected uncertainty in an individual measurement. In other words, if we measure x again, we will usually find it lies within $\pm \delta x$ of the mean.

There are two sources of this uncertainty. One is the instrumental resolution, often called the reading error. The other is uncertainty inherent in the procedure. If you use a ruler

whose smallest division is 1 mm, you can guess reliably to about $\delta x = \pm 0.1$ mm in a measurement of an object with smooth, regular edges. On the other hand, if you are attempting to measure the length of a rough, stretchy object, like your shirt sleeve, your uncertainty would increase to about $\delta x = \pm 1.0$ mm or more. The uncertainty can never be smaller than the instrumental resolution, and often it is larger. Only by repeating a measurement many times can you be sure of the uncertainty.

Of course, the uncertainty in the mean should be smaller than the uncertainty in a single reading. We will learn in a later lab that the uncertainty in the mean is:

$$\delta \bar{x} = \frac{\delta x}{\sqrt{N}}$$

Part III: Significant Figures

The precision of an experimental result can be inferred from the number of digits, or significant figures, recorded in the result. Use the following rules for significant figures when reporting your results:

- The leftmost non-zero digit is the most significant digit.
- If there is no decimal point, the rightmost non-zero digit is the least significant digit.
- If there is a decimal point, the rightmost digit is the least significant digit, even if it is zero.
- All digits between the least and most significant digits are counted as significant digits

The following examples illustrate these rules:

Example 1: The following numbers each have four significant digits:

1234	1000.
123,400	10.30
123.4	0.0001230
1004	100.0

If there is an ambiguity, use *scientific notation*. For example, to report 1000 with three significant digits, write it as 1.00×10^3 .

Example 2: In multiplication or division it is usually acceptable to keep the same number of significant figures in the product or quotient as are in the **least** precise factor.

$$2.6 \times 31.7 = 82.42 \rightarrow 82$$

$$5.3 \div 748 = 0.007085 \rightarrow 0.0071$$

Example 3: In addition and subtraction, the number of significant digits in the result is

governed by where the rightmost digit appears in all the numbers. The result is rounded to the most significant uncertain digit. Handling of significant figures in addition and subtraction are illustrated with the examples below:

$$51.4 - 1.67 = 49.73 \rightarrow 49.7$$

$$7146 - 12.8 = 7133.2 \rightarrow 7133$$

$$20.8 + 18.72 + 0.851 = 40.371 \rightarrow 40.4$$

$$1.4693 + 10.18 + 1.062 = 12.7113 \rightarrow 12.71$$

Note the following rule to be used in this class in reporting an uncertainty:

Experimental error will always be stated (rounded to) one significant digit

Please look carefully at the following examples:

Measurement	Uncertainty	Reported Value
453.79 g	0.5 g	453.8 ± 0.5 g
37546 mi	27 mi	(37.55 ± 0.03) × 10 ³
256.8 hr	5.1 hr	257 ± 5 hr

Note the following on exact numbers:

There is no uncertainty in an exact or an accepted value

For example, in the expression $C = 2\pi R$, the 2 and the π have no uncertainty.

You must be a bit careful with this rule. There are very few physical constants for which the values are known exactly. Some physical constants have defined values; hence, zero uncertainty is associated with them. The vast majority of physical constants are experimentally determined. Hence, even these quantities have uncertainties. For example, the speed of light was in 1983 assigned the exact value of 2.99792458×10^8 m/s. There is no error! The gravitational constant G , on the other hand, is experimentally determined to be $G = 6.6725985 \times 10^{-11}$ m³/kg·s² ± 128 parts per million. The 128 parts per million uncertainty puts the error out at the digits(85) in the value of G . In any given typical lab setting a value of 6.67×10^{-11} m³/kg·s² would be used. However, as you can see, there are three more digits beyond the 7 before the first uncertain digit (8) arises. Thus, the value $G = 6.67 \times 10^{-11}$ m³/kg·s² can be treated as exact if the least certain variable in the rest of the equation has three or fewer significant digits. If the least certain variable has four significant figures, then you could treat G as exact by using $G = 6.673 \times 10^{-11}$ m³/kg·s². The same is true for most constants you will use in this course.

Part IV: Fractional and Absolute Uncertainty

Suppose the reported measurement for the side of a cube is $L = 8.5 \pm 0.2$ cm (i.e., $\delta L = 0.2$ cm). Suppose you also measure the height of a skyscraper as $H = 15572.8 \pm 0.2$ cm (i.e., $\delta H = 0.2$ cm). Here are two measurements with the same absolute uncertainty, yet somehow the skyscraper seems to be a much more precise measurement.

The fractional uncertainty reveals the difference. For a measurement or calculation result, reported as $f \pm \delta f$, the fractional uncertainty is given as $\frac{\delta f}{f}$: So, for the cube:

$$\frac{\delta L}{L} = \frac{0.2}{8.5} = 0.024$$

and for the skyscraper,

$$\frac{\delta H}{H} = \frac{0.2}{15572.8} = 0.00013$$

which is a very small fractional error. Often fractional errors are quoted in percent, so we see for the cube

$$\frac{\delta L}{L} = 0.024 = 2.4\%$$

We may also report fractional rather than absolute uncertainty, for example, $L = 8.5$ cm \pm 2% (round to one digit).

Part V: Propagation of Uncertainty in Calculations

There are two methods of propagating the uncertainties of variables used in an equation to the uncertainty of the result. The simplest is to add the fractional errors. For example, to calculate the volume of a rectangular prism, we would use the equation:

$$V = lwh$$

then, the fractional error is:

$$\frac{\delta V}{V} = \frac{\delta l}{l} + \frac{\delta w}{w} + \frac{\delta h}{h}$$

The problem with this is that it overestimates the uncertainty, because it assumes all of the errors in l , w , and h are in the same direction (i.e. all were measured too large or too small). Usually the error in each variable is independent of the error in every other variable, so it ends up that some of the errors will cancel. The correct mathematics to add the uncertainties in this case comes from the problem called the “random walk.” The random walk problem gives a very elegant expression that treats uncertainty in functions of multiple variables in an experimental setting. To use this, we need one more mathematical tool: the partial derivative.

If $f = f(x,y,z)$ is a function of three variables x , y , and z , we write:

$$\partial f / \partial x = \text{partial derivative of } f \text{ with respect to } x$$

$\partial f / \partial x$ is the result of differentiating f with respect to x , treating y and z as constants. It is the same as the single derivative you are accustomed to, while pretending that y and z are constants.

The general rule for determining the uncertainty δf when f is a function of several variables, $f = f(x,y,\dots,z)$, and assuming independent and random uncertainties in x , y , ..., z is:

$$\delta f = \sqrt{\left(\frac{\partial f}{\partial x} \delta x\right)^2 + \left(\frac{\partial f}{\partial y} \delta y\right)^2 + \left(\frac{\partial f}{\partial z} \delta z\right)^2 + \dots}$$

In words, this equation states that the uncertainty in f is the square root of the sum of the squares of the uncertainties in the variables weighted by their effect on f . This equation will be used for all of the error propagation done in this course. Here are some examples of how this works:

Example 1: Suppose we have a rectangular box whose dimensions we measure to be:

	Value	Error
Height, h	4.25 m	0.03 m
Width, w	3.15 m	0.03 m
Length, l	9.47 m	0.08 m

What is the volume and its uncertainty? First, note the following:

$$\delta h = 0.03, \delta w = 0.03, \delta l = 0.08$$

$$V = lwh, \frac{\partial V}{\partial h} = lw, \frac{\partial V}{\partial w} = lh, \frac{\partial V}{\partial l} = wh$$

Therefore,

$$V = lwh = 4.25 \times 3.15 \times 9.47 = 126.779625 \text{ m}^3$$

$$\begin{aligned} \delta V &= \sqrt{\left(\frac{\partial V}{\partial h} \delta h\right)^2 + \left(\frac{\partial V}{\partial w} \delta w\right)^2 + \left(\frac{\partial V}{\partial l} \delta l\right)^2} = \sqrt{(lw\delta h)^2 + (lh\delta w)^2 + (wh\delta l)^2} \\ &= \sqrt{(3.15\text{m} \times 9.47\text{m} \times 0.03\text{m})^2 + (9.47\text{m} \times 4.25\text{m} \times 0.03\text{m})^2 + (3.15\text{m} \times 4.25\text{m} \times 0.08\text{m})^2} \\ &= \sqrt{3.4058\text{m}^6} = 1.845 \text{ m}^3 \end{aligned}$$

To report the result properly we must have the correct number of significant figures in the answer and the uncertainty must have the proper precision. Thus, after rounding, we report:

$$V = 127 \pm 2 \text{ m}^3$$

Example 2: A jogger runs at a rate $R = 128 \pm 2$ steps/min and each step length $L = 45 \pm 2$ in. How long will it take for the jogger to run one lap on a circular track of diameter $D = 200. \pm 1$ ft.? (Recall that: $t = s/v$; $s = \pi D$; $v = RL$ where $t =$ time, $s =$ total distance, and $v =$ velocity). So, we see that

$$t = \frac{\pi D}{RL} = \frac{\pi \times 2400 \text{ in}}{128 \text{ step/min} \times 45 \text{ in/step}} = 1.308911 \text{ min}$$

$$\frac{\partial t}{\partial R} = -\frac{\pi D}{R^2 L}, \quad \frac{\partial t}{\partial L} = -\frac{\pi D}{L^2 R}, \quad \frac{\partial t}{\partial D} = \frac{\pi}{RL}$$

Therefore,

$$\delta t = \sqrt{\left(\frac{\partial t}{\partial R} \delta R\right)^2 + \left(\frac{\partial t}{\partial L} \delta L\right)^2 + \left(\frac{\partial t}{\partial D} \delta D\right)^2} = \sqrt{\left(-\frac{\pi D}{R^2 L} \delta R\right)^2 + \left(-\frac{\pi D}{L^2 R} \delta L\right)^2 + \left(\frac{\pi}{RL} \delta D\right)^2}$$

Note that we can factor $t = \frac{\pi D}{RL}$ out of the radical. Therefore, the statement simplifies to

$$\delta t = t \sqrt{\left(\frac{\delta R}{R}\right)^2 + \left(\frac{\delta L}{L}\right)^2 + \left(\frac{\delta D}{D}\right)^2} = (1.308911 \text{ min}) \sqrt{\left(\frac{2 \text{ in}}{128 \text{ in}}\right)^2 + \left(\frac{2 \text{ in}}{45 \text{ in}}\right)^2 + \left(\frac{1 \text{ ft}}{200 \text{ ft}}\right)^2}$$

$$\delta t = 6.201 \times 10^{-2} \text{ min}$$

So we report the result as:

$$t = 1.31 \pm 0.06 \text{ min}$$

Example 3: The acceleration due to gravity, g , can be determined from the motion of a pendulum according to the equation:

$$g = \frac{4\pi^2 L}{T^2}$$

where $L =$ length of the pendulum and $T =$ period. Suppose the period and length are measured to be $T = 2.015 \pm 0.005$ s and $L = 1.001 \pm 0.008$ m, respectively. What is the value of and uncertainty in g ?

Note that

$$\frac{\partial g}{\partial L} = \frac{4\pi^2}{T^2}, \quad \frac{\partial g}{\partial T} = -\frac{8\pi^2 L}{T^3}$$

Therefore,

$$\delta g = \sqrt{\left(\frac{\partial g}{\partial L} \delta L\right)^2 + \left(\frac{\partial g}{\partial T} \delta T\right)^2} = \sqrt{\left(\frac{4\pi^2}{T^2} \delta L\right)^2 + \left(-\frac{8\pi^2 L}{T^3} \delta T\right)^2} = 0.092 \frac{\text{m}}{\text{s}^2}$$

Our final answer is, rounding our error to 1 significant digit,

$$g = 9.73 \pm 0.09 \frac{\text{m}}{\text{s}^2}$$

Part VI: Homework Problems

Answer the following questions and turn in at the beginning of lab next week.

Q1: [0.5 pts] Using the model of the shot pattern about a target, can you say anything about the accuracy of the set of data points given in Table 1? What about the precision?

Q2: [2 pts] Measured values and their errors are given in each exercise, and you are asked to calculate a quantity and its error. **You will be expected to follow this same format when doing labs and experiments in this lab.**

- A. The measured values are $d_1 = 3.42 \pm .01$ m and $d_2 = 7.65 \pm .05$ m. Defining $d = d_2 - d_1$, calculate d and its error.

d =

- B. If $d_1 = 6.710 \pm .005$ m, $d_2 = 0.64 \pm 0.02$ m, and $d_3 = 1.71 \pm 0.01$ m and defining $d = d_1 + d_2 - d_3$, then calculate the error.

d =

- C. The radius of a circle is measured to be 6.072 ± 0.005 cm. Calculate the area of the circle and its error assuming π is known to an arbitrary great accuracy.

A =

- D. The distance an object travels and the time taken are measured to be $d = (1.748 \pm 0.001) \times 10^{-2}$ m and $t = (5.41 \pm 0.05) \times 10^{-3}$ s. Calculate the speed v and its error where $v = d/t$.

v =

Q3: [1 pt] Rewrite the following answers in their clearest forms, with a suitable number of significant figures.

- A. Measured height = $5.03 \pm .04329$ meters.

- B. Measured time = 19.5432 ± 1 sec.

- C. Measured charge = $-3.21 \times 10^{-19} \pm 2.67 \times 10^{-20}$ coulombs.

- D. Measured wavelength = $0.000000563 \pm 0.00000007$ meters.

- E. Measured momentum = $3.267 \times 10^{-3} \pm 175$ gm cm/sec.

Q4: [1.5 pts] In an experiment to check conservation of angular momentum, a student obtains the results shown in the table below for the initial and final angular momenta (L and L' , respectively) of a rotating system. Using the below data, calculate the value of $\Delta L = L - L'$, as well as the error $\delta(\Delta L)$. For each set of data, are the student's results consistent with conservation of angular momentum (i.e. does ΔL equal zero within experimental uncertainty)?

L	L'	$\Delta L = L - L'$	$\delta(\Delta L)$	Conserved?
3.0 ± 0.3	32.7 ± 0.6			
7.4 ± 0.5	8.0 ± 0.1			
14.3 ± 0.1	16.5 ± 0.1			
25 ± 2	24 ± 2			
32 ± 3	31 ± 3			
37 ± 2	41 ± 2			

Q5: [2 pts] In order to calculate the acceleration of a cart, a student measures its initial and final velocities, v_i and v_f respectively, and computes the difference, $\Delta v = v_f - v_i$; the data in two separate trials (all in cm/sec) are shown in the table below. **All four measurements have 1% accuracy.**

- A. Calculate the absolute uncertainties in all four measurements. Find Δv and its uncertainty $\delta(\Delta v)$ in each run.
- B. Compute the percentage uncertainty for each of the two values of Δv . (Note: your answers here, particularly for the second run, illustrate the potentially disastrous results of measuring a small number by taking the difference of two much larger numbers.)

	v_f	v_i	δv_f	δv_i	Δv	$\delta(\Delta v)$	$\delta(\Delta v)$ as %
1st Run	14.0	18.0					
2nd Run	19.0	19.6					

Q6: [1.5 pts] In an actual pendulum experiment, similar to the one described in Part V, Example 3, an experimenter typically measures the period T for various lengths L and computes g for each. The following table gives a set of data for 2 separate pendulum experiments. The uncertainties in measuring length and time are ± 0.5 cm and ± 0.05 s, respectively.

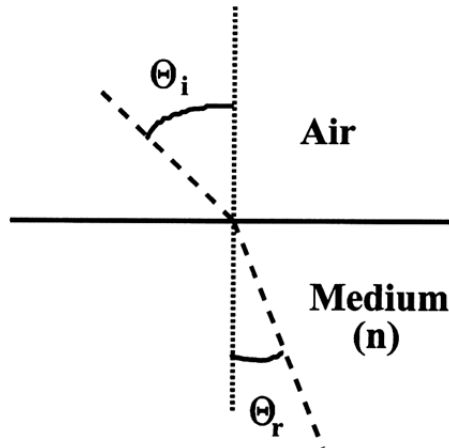
Calculate the missing values for the second set of data displayed in the table.

	L (in cm)	T (in s)	g ($\frac{cm}{s^2}$)	$\delta L/L$	$\delta T/T$	$\delta g/g$	Answer for g
1st Run	93.8	1.944	980	0.0053	0.026	0.052	$9.8 \pm 0.5 \frac{m}{s^2}$
2nd Run	80.3	1.761					

Q7: [1.5 pts] According to Snell's law, when a ray of light passes from air into a transparent material medium (see figure), the following relationship holds:

$$n \sin \theta_r = \sin \theta_i$$

Where n is the index of refraction of the medium.



For the data set shown below, compute n , δn , and $\delta n/n$ for each trial. **Remember that the error in Θ will be in radians!**

	θ_r	θ_i	n	δn	$\delta n/n$
1st Trial	$13^\circ \pm 0.1^\circ$	$20^\circ \pm 0.1^\circ$			
2nd Trial	$29^\circ \pm 0.1^\circ$	$50^\circ \pm 0.1^\circ$			