Midterm 1

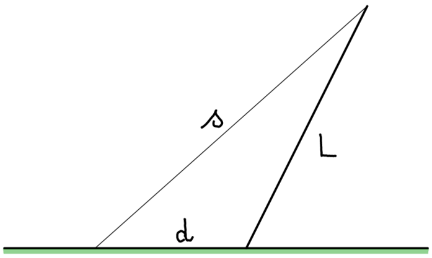
(I) *[25 pts.]* Consider the relation $a = \frac{b\,c}{4d} + f$ in which a is a distance, b is a velocity, and d is an acceleration. What would be the SI units to be used when specifying the values of f and c?

- Variable f is of the same nature as a which is a distance so f could be expressed in $\boxed{m}$

- From the equation $[c] = \left[\frac{a\,d}{b}\right] = \frac{\cancel{L} \times L T^{\cancel{-2}\,-1}}{\cancel{L} T^{\cancel{-1}}} = L T^{-1}$ so c is a velocity

Variable c could be expressed in $m.s^{-1}$

(II) *[25 pts.]* A post of length L is held by a cable of length s tied to the top of the post and to a point on the ground at a distance d from the foot of the post as shown on the figure. Express the angle between the post and the horizontal direction.



Let θ be the angle the post & the horizontal direction. Let h be the height of the top of the post above ground & x the distance of the perpendicular projection of the top of the post on the ground from the foot of the post. Then we have

$s^2 = (d+x)^2 + h^2$, $L^2 = x^2 + h^2$, and $\cos\theta = \frac{x}{L}$

The second equation $h^2 = L^2 - x^2$ can be used in the first. $s^2 = (d+x)^2 + L^2 - x^2 = d^2 + 2dx + \cancel{x^2} + L^2 - \cancel{x^2}$ so $x = \frac{s^2 - L^2 - d^2}{2d}$

so using this in the third equation $\cos\theta = \frac{s^2 - L^2 - d^2}{2dL}$

so $\boxed{\theta = \cos^{-1}\left[\frac{s^2 - L^2 - d^2}{2dL}\right]}$

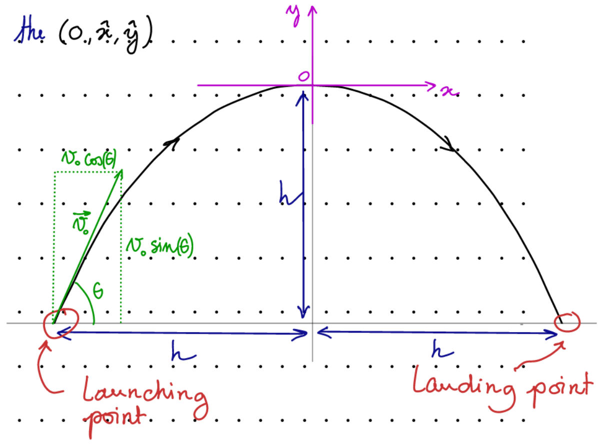
If $s = L + d$, $\theta = \cos^{-1}(1) = 0$ which seems right
if the post is vertical then $s^2 = L^2 + d^2$ & $\theta = \cos^{-1}(0) = \frac{\pi}{2}$ which is Consistent.

(III) *[25 pts.]* You are driving your car along a straight road with a velocity of 72 km/h when suddenly an obstacle appears 80 m in front of you. Assuming uniformly accelerated motion, how much of an average deceleration must you apply to the car so that it comes to a stop just before reaching the obstacle?

Let the initial velocity of the car be $v_0 = 72\,\text{km/h} = 20\,\text{m s}^{-1}$. With the magnitude of the average acceleration during the breaking phase noted a, the car comes to a stop after a time $t_s = \frac{v_0}{a}$. From the beginning of the breaking phase & until the car comes to a stop, the car moves by $d = v_0 t_s - \frac{1}{2} a t_s^2 = \frac{1}{2}\frac{v_0^2}{a}$ & we need $d < D$ with $D = 80\,\text{m}$ the distance at which the obstacle initially appeared. So $\frac{1}{2}\frac{v_0^2}{a} < D$ or $\boxed{a > \frac{1}{2}\frac{v_0^2}{D}}$

Numerically $a > \frac{1}{2}\frac{(20\,\text{m s}^{-1})^2}{80\,\text{m}}$ or $\boxed{a > 2.5\,\text{m s}^{-2}}$

(IV) *[25 pts.]* A system is designed to launch a projectile with an initial velocity v_0. Once airborne, the projectile has a uniform downward acceleration g. Find an expression for the angle of the launch direction from horizontal such that the projectile culminates at an elevation h higher than the launching point, and lands at a horizontal distance $2h$ from the launching point.



- Let the angle between the launch direction & horizontal be θ. & let's use the $(O, \hat{x}, \hat{y})$ coordinate system attache to the point of culmination as in the drawing. And let's also choose time $t = 0$ at culmination.

- The initial velocity is $\vec{v}_0 = v_0 \cos\theta\, \hat{x} + v_0 \sin\theta\, \hat{y}$ & since the gravitational acceleration $\vec{g} = -g\hat{y}$ has no x component, the x component of the velocity must be constant & equals $v_0 \cos(\theta)$. With our choice of coordinates the position vector or the projectile is

$$\vec{r} = v_0 \cos(\theta)\, t\, \hat{x} - \frac{1}{2} g t^2 \hat{y} = r_x \hat{x} + r_y \hat{y}$$

and its velocity is $\vec{v}(t) = v_0 \cos(\theta) \hat{x} - g t \hat{y}$

- The angle θ is such that when $r_x = +h$, $r_y = -h$. let t_h be the time when this happens

→ $r_x = h \Rightarrow v_0 \cos(\theta) t_h = h$ or $t_h = \dfrac{h}{v_0 \cos(\theta)}$

→ $r_y = -h \Rightarrow -\frac{1}{2} g t^2 = -h$ or $\frac{1}{2} g \dfrac{h^{\cancel{2}}}{v_0^2 \cos^2\theta} = \cancel{h}$ & $2\cos^2(\theta) = \dfrac{gh}{v_0^2}$

→ also $\vec{v}(t_h) = v_0 \cos(\theta) \hat{x} - g \dfrac{h}{v_0 \cos(\theta)} \hat{y}$ while we know that the y component at the velocity is opposit to the y component of the launch velocity so $v_0 \sin\theta = g \dfrac{h}{v_0 \cos(\theta)}$ or $\dfrac{gh}{v_0^2} = \sin(\theta)\cos(\theta)$

→ but we found also that $\dfrac{gh}{v_0^2} = 2\cos^2(\theta)$ so $\sin(\theta)\cancel{\cos(\theta)} = 2\cos^{\cancel{2}}(\theta)$

and $\dfrac{\sin(\theta)}{\cos(\theta)} = \tan(\theta) = 2$ or $\boxed{\theta = \tan^{-1}(2) \simeq 1.107\ldots \text{ rad} \simeq 63.4^\circ \ldots}$