1) **Cosmic Neutron**

The Telescope Array experiment in Millard county Utah can detect cosmic rays with energies up to \( E = 10^{19} \text{eV} \). We assume that particles detected at such high energy are neutrons, which have a mass \( m_n = 939 \text{MeV}/c^2 \) and a proper life time \( \tau = 885 \text{s} \).

a) (7 points) What is the Lorentz factor \( \gamma \) of these high energy neutrons?

\[
\gamma = \frac{E}{mc^2} = \frac{10^{19} \text{eV}}{939 \times 10^6 \text{eV}} = 1.065 \times 10^{10}.
\]

b) (7 points) The proper diameter of the Earth is 12,756km. As it would be observed from the neutron, what is the diameter of the Earth along the direction of motion of the neutron?

Length contraction \( L' = \frac{L}{\gamma} = \frac{12756 \text{km}}{1.065 \times 10^{10}} = 1.19 \times 10^{-6} \text{km} \)

or \( L' = 1.2 \text{mm} \).
c) (7 points) How far may these high energy neutrons may be coming from? (Hint: they reach the Earth within a proper time shorter than their proper life time)

* As seen from Earth, the life time of the neutron is \( \tau T \).
  
  It is so energetic it can be considered to go at the speed of light so the distance it travels is \( D = \gamma c \tau \).

  Numerically \( D = 1.06 \times 10^9 \times 885 \times 310^8 = 2.8 \times 10^{21} m \)
  
  which corresponds to \( 2.36 \times 10^6 \) light years.

Note: If you do not see the approximation \( v = c \) right away, you can always do \( D = v \gamma c \tau \) or \( D^2 = v^2 \gamma^2 c^2 \tau^2 = \frac{v^2 \gamma c^2 \tau^2}{c^2} \).

\( \gamma^2 = \frac{1}{1 - \frac{v^2}{c^2}} \) so \( \frac{v^2}{c^2} = 1 - \frac{1}{\gamma^2} \) and \( D^2 = (1 - \frac{1}{\gamma^2}) \gamma^2 c^2 \tau^2 \)

or \( D = \sqrt{\gamma^2 - 1} c \tau \) and when \( \gamma^2 \gg 1 \), \( D \approx \gamma c \tau \) as used above.
2) **Lorentz transformation**

Electrons \( m_e = 511 \text{keV}/c^2 \) at rest are accelerated by an electric potential difference \( V = 300 \text{kV} \) along direction \(+y\). Protons are moving at half the speed of light along direction \(+x\) making a right angle with direction \(+y\).

a) *(7 points)* Write the energy-momentum four-vector of the electron in the laboratory

\[
\begin{align*}
K_e &= 300 \text{ keV} \\
E &= K_e + m_e c^2 = 811 \text{ keV} \\
\mathbf{p}_e c &= (E^2 - m_e^2 c^4)^{1/2} = (811^2 - 511^2) \text{keV} = 629 \text{ keV}.
\end{align*}
\]

The energy-momentum reads

\[
\begin{pmatrix}
E \\
\mathbf{p}_x c \\
\mathbf{p}_y c \\
\mathbf{p}_z c
\end{pmatrix} =
\begin{pmatrix}
811 \text{ keV} \\
0 \\
629 \text{ keV} \\
0
\end{pmatrix}.
\]

b) *(7 points)* Using a Lorentz transformation, calculate the energy-momentum four-vector of the electron from the point of view of the proton.

\[
\begin{pmatrix}
E' \\
\mathbf{p}_x' c \\
\mathbf{p}_y' c \\
\mathbf{p}_z' c
\end{pmatrix} =
\begin{pmatrix}
\gamma & -\beta \gamma & 0 & 0 \\
-\beta \gamma & \gamma & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
E \\
\mathbf{p}_x c \\
\mathbf{p}_y c \\
\mathbf{p}_z c
\end{pmatrix} =
\begin{pmatrix}
\gamma E - \beta \gamma P_x c \\
-\beta \gamma E + \gamma P_x c \\
0 \\
0
\end{pmatrix}.
\]

Here \( \beta = 0.5 \) ; \( \gamma = 1.155 \)

\[
\begin{pmatrix}
E' \\
\mathbf{p}_x' c \\
\mathbf{p}_y' c \\
\mathbf{p}_z' c
\end{pmatrix} =
\begin{pmatrix}
1.155 \times 811 \text{ keV} \\
0.577 \times 811 \text{ keV} \\
629 \text{ keV} \\
0
\end{pmatrix} =
\begin{pmatrix}
936 \text{ keV} \\
468 \text{ keV} \\
629 \text{ keV} \\
0
\end{pmatrix}.
\]
c) \((7 \text{ points})\) From the point of view of the proton, what is the kinetic energy of the electron?

\[
K_e' = E' - mc^2 = (3.6 - 5.11) \text{ keV} = 4.25 \text{ keV}.
\]

d) \((7 \text{ points})\) From the point of view of the proton, what is the angle between the \(x\) direction and the direction of motion of the electron?

\[
\theta = \arcsin \left[ \frac{-468}{\sqrt{468^2 + 623^2}} \right] = 126.6^\circ \quad \theta
\]
3) **Black-body radiation**

The filament of a light bulb is \( l = 20\text{mm} \) long, it has a radius \( r = 0.05\text{mm} \) and it is maintained at a temperature \( T = 5000\text{K} \) by an electric current. We assume the black body radiation is emitted isotropically. At night, you observe the light bulb from a distance \( D = 10\text{km} \) with the pupil of your eye fully dilated to a radius \( \rho = 3\text{mm} \).

(Data: \( \sigma = 5.6 \times 10^{-8} \text{W} \cdot \text{m}^{-2} \cdot \text{K}^{-4} \), \( b = 2.9 \times 10^{-3} \text{m} \cdot \text{K} \), \( h = 6.62 \times 10^{-34} \text{J} \cdot \text{s} \)).

a) (7 points) What is the power radiated by the filament?

\[
P = A \sigma T^4 = 2\pi \pi r l \sigma T^4
\]

\[
P = 2\pi \times 5 \times 10^{-5} \times 20 \times 10^{-3} \times 5.6 \times 10^{-8} \times 5000^4
\]

\[
P = 220 \text{W}
\]

b) (7 points) What is radiation power entering your eye?

\[
P_{\text{eye}} = \frac{\pi \rho^2}{4\pi D^2} P = \frac{(3 \times 10^{-3})^2}{4 \times (10^4)^2} \times 220 = 4.95 \times 10^{-12} \text{W}.
\]
c) \( (7 \text{ points}) \) What is the wavelength at which the power radiated per wavelength interval is the greatest?

\[
\lambda_{\text{max}} = \frac{b}{T} = \frac{2.9 \times 10^{-3} \text{ m} \cdot \text{k}}{5000 \text{ k}} = 580 \text{ nm}.
\]

d) \( (7 \text{ points}) \) How many photons are entering your eye every second? As an approximation, you will consider all the power to be radiated with a wavelength \( \lambda = 600 \text{ nm} \).

\[
J = \frac{\text{Power} \cdot \lambda}{hc} = \frac{4.95 \times 10^{-12} \times 600}{6.62 \times 10^{-34} \times 3 \times 10^8} = 1.5 \times 10^{16} \text{ photons/s}.
\]
4) **Heisenberg uncertainty relation**

a) (7 points) In 1983, physicists working at the CERN, observed the \( Z_0 \), a particle that had been postulated in 1968 by physicists Weinberg, Salam and Glashow in their theory of the weak interaction. The \( Z_0 \) article was found to have an average mass \( m_{Z_0} = 91.2 \text{GeV/c}^2 \) with a standard deviation \( \Delta m_{Z_0} = 2.6 \text{GeV/c}^2 \). Estimate the life time in seconds of the \( Z_0 \) particle (Data: \( h = 6.62 \times 10^{-34} \text{J} \cdot \text{s} \), \( e = 1.6 \times 10^{-19} \text{C} \)).

\[
\Delta E \Delta t \approx \frac{\hbar}{c} \\
\Delta t \approx \frac{\hbar}{\Delta m c^2} = \frac{6.62 \times 10^{34}}{2\pi \times 2.6 \times 10^9 \times 1.6 \times 10^{-19}} = 2.5 \times 10^{-25} \text{s}
\]

b) (7 points) Electrons (\( m_e = 511\text{keV/c}^2 \)) are moving along the \( x \) direction in a periodic potential of period \( L \). The potential difference between a potential crest and a potential trough is \( \Delta V = 0.4\text{eV} \). Estimate the minimal period \( L \) so electrons of low enough energy may be trapped in one valley (Data: \( h = 6.62 \times 10^{-34} \text{J} \cdot \text{s} \), \( e = 1.6 \times 10^{-19} \text{C} \)).

\[
\Delta P \ L \gg \frac{\hbar}{2} \Rightarrow \ L \gg \frac{\hbar}{2 \sqrt{2mE}} \\
L \gg \frac{6.62 \times 10^{34}}{4\pi \sqrt{2 \times 511 \times 10^3 \times 0.4 \times (1.6 \times 10^{-19})^2 / (3 \times 10^8)^2}} = 1.5 \times 10^{-10} \text{m} -
\]
5) **Down the potential step** (note: questions a,b and c are independent from each other)
We consider a potential energy $V(x) = 0$ for $x \leq 0$ (region I) and $V(x) = -V_0$ for $x > 0$ (region II). For states of energy $E > 0$, the time independent Schrödinger equation for particles of mass $m$ reads $\frac{d^2 \psi}{dx^2} = -k^2 \psi$ in region I and $\frac{d^2 \psi}{dx^2} = -k_0^2 \psi$ in region II with $k_i = \sqrt{2mE}/\hbar$ and $k_0 = \sqrt{2m(E + V_0)}/\hbar$. The general form of the solution is $\psi(x) = Ae^{ik_i x} + Be^{-ik_i x}$ in region I and $\psi(x) = Ce^{ik_0 x} + De^{-ik_0 x}$ in region II. The time dependent wave function can be written $\Psi(x,t) = \psi(x)e^{-iEt/\hbar}$. A beam of particles is shot from region I toward region II.

a) (7 points) Explain why we can set $D=0$?

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b) (7 points) Write the boundary condition equations in $x = 0$.

$$
\Psi_I(0) = \Psi_{II}(0) \quad \text{or} \quad A + B = C
$$

$$
\frac{d\Psi_I}{dx}(0) = \frac{d\Psi_{II}}{dx}(0) \quad \text{or} \quad k_1 A - k_2 B = k_2 C
$$
d) (7 points) The wave of amplitude $A$ corresponds to the beam of particles shot toward the potential step. The wave of amplitude $B$ corresponds to the particles reflected by the potential step. The wave of amplitude $C$ corresponds to the particles transmitted through the potential step. Express $B$ and $C$ in terms of $A$, $k_i$ and $k_\|$

$$k_\| A - k_\perp B = k_\perp A + k_\| B$$

$$B = \frac{k_\| - k_\perp}{k_\| + k_\perp} A$$

$$C = \left(1 + \frac{k_\| - k_\perp}{k_\| + k_\perp}\right) A = \frac{2k_\|}{k_\| + k_\perp} A$$

e) (7 points) A wave of amplitude $A$ and wave vector $k_i$ corresponds to a beam intensity (particles per seconds) $I_A=\hbar k_i A^2/m$. In the same way $I_B=\hbar k_i B^2/m$ and $I_C=\hbar k_\| C^2/m$. Express the reflection and transmission coefficients $R=I_B/I_A$ and $T=I_C/I_A$. Verify they add up to 1.

$$R = \frac{I_B}{I_A} = \frac{k_\| B^2}{k_\| A^2} = \left(\frac{k_\| - k_\perp}{k_\| + k_\perp}\right)^2$$

$$T = \frac{I_C}{I_A} = \frac{4 k_\|^2 k_\perp}{k_\| \left(k_\| + k_\perp\right)^2}$$

$$R + T = \frac{4 k_\| k_\perp + (k_\| - k_\perp)^2}{(k_\| + k_\perp)^2} = 1$$

c) (7 points) Draw the representation of a real (not complex) wave passing the potential step. Are there bound states?
$R_I < R_{II}$