# Derivation of the Lorentz transformation from a minimal set of assumptions

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## 1 Introduction

Originally, the Lorentz transformation was derived from the following assumptions [1]:
• Linearity
• Invariance of $c$, the speed of light
• Existance of a composition law
• Existance of a neutral element
• Reflection invariance

And, in the classroom, this also is how we derive it in general. In 1976 however, Levy-Leblond reported a derivation which did not make use of the invariance of the speed of light [2]. The limit speed which appears in the derivation is identified, a-posteriori, to the speed of light. Even more recently, the Lorentz transformation was derived from the following, even lighter, set of assumptions:
• Linearity
• Internality of the composition law
• Reflection invariance
These hypotheses are not sufficient by themselves to define a group but they turn out to be sufficiently constraining to imply the Lorentz transformations and their full group structure. Note also that these assumptions are in fact expressions of the relativity principle. Linearity ensures the inverse transformation also is linear[1]. Internality of composition ensures only the relative reference frame characteristics enter the transformation laws. Finally, reflection invariance ensures the form of the transformation do not depend on specifics of the choice of coordinate systems attached to each reference frame. Here, we proceed with this derivation which was first published by L. Nottale in [3].

*after an original work by L. Nottale [3]
[1] As recently reminded to me by Eugene Mishchenko

# 2 The derivation

## 2.1 Linearity

From coordinates $(t, x)$ in one reference frame, we wish to obtain the coordinates $(t', x')$ in another in motion at speed $u$ with respect to the first one. [2]

$$\begin{aligned} x' &= a(u)x - b(u)t \quad = a(u)(x - \tfrac{b(u)}{a(u)}t) \\ t' &= \alpha(u)t - \beta(u)x \end{aligned} \tag{1}$$

Defining velocity as $u = b/a$, without any loss of generality, we can write

$$\begin{aligned} x' &= \gamma(u)(x - ut) \\ t' &= \gamma(u)(A(u)t - B(u)x) \quad = \gamma(u)A(u)(t - \tfrac{B(u)}{A(u)}x) \end{aligned} \tag{2}$$

## 2.2 Reflection invariance

Inverting the orientation of both the $x$ and $x'$ axis we obtain the following transformation

$$\begin{aligned} -x' &= \gamma(\tilde{u})(-x - \tilde{u}t) \\ t' &= \gamma(\tilde{u})(A(\tilde{u})t + B(\tilde{u})x) \end{aligned} \tag{3}$$

which has to be equivalent to the transformation given in equation (2) implying that $\tilde{u} = -u$ and

$$\gamma(-u) = \gamma(u) \tag{4}$$

$$A(-u) = A(u) \tag{5}$$

$$B(-u) = -B(u) \tag{6}$$

## 2.3 Internal composition

Applying successively two transformations:

$$\begin{aligned} x' &= \gamma(u)(x - ut) \\ t' &= \gamma(u)(A(u)t - B(u)x) \\ x'' &= \gamma(v)(x' - vt') \\ t'' &= \gamma(v)(A(v)t' - B(v)x') \end{aligned} \tag{7}$$

gives

$$\begin{aligned} x'' &= \gamma(u)\gamma(v)[1 + B(u)v][x - \tfrac{u+vA(u)}{1+vB(u)}t] \\ t'' &= \gamma(u)\gamma(v)[A(u)A(v) + uB(v)][t - \tfrac{A(u)B(v)+B(v)}{A(u)A(v)+uB(v)}x] \end{aligned} \tag{8}$$

which can be identified to a single transformation of parameter $w$ with the following relationships

$$w = \frac{u + vA(u)}{1 + vB(u)} \tag{9}$$

[2] We restrict ourselves to a single spatial dimension.

$$\gamma(w) = \gamma(u)\gamma(v)[1 + vB(u)] \tag{10}$$

$$\gamma(w)A(w) = \gamma(u)\gamma(v)[A(u)A(v) + uB(v)] \tag{11}$$

$$\frac{B(w)}{A(w)} = \frac{A(v)B(u) + B(v)}{A(u)A(v) + uB(v)} \tag{12}$$

### 2.4 Finding $A(u)$, $B(u)$ and $\gamma(u)$

• Taking the ratio (11)/(10) and replacing $w$ by the expression in (9) we get

$$A\left(\frac{u + vA(u)}{1 + vB(u)}\right) = \frac{A(u)A(v) + uB(v)}{1 + vB(u)} \tag{13}$$

• With $v = 0$ this become

$$A(u) = A(u)A(0) + uB(0) \text{ or } A(u)[1 - A(0)] = uB(0) \tag{14}$$

in which $u = 0$ implies $A(0) = 0$ or $A(0) = 1$

• If $A(0) = 0$, (14) becomes $A(u) = uB(0)$ but we know that $B(0) = 0$, so $A(u) = 0$ and equation 12 becomes $A(w) = uB(w)$ so $B(w) = 0$ and looking back at (2) $\forall u$, we have $t' = 0$. It is a completely degenerated case which we can exclude. Hence $A(0) = 1$.

• Let us set $v = -u$ in equation (13)

$$A\left(\frac{u - uA(u)}{1 - uB(u)}\right) = \frac{A(u)A(-u) + uB(-u)}{1 - uB(u)} \tag{15}$$

or

$$A\left(\frac{u - uA(u)}{1 - uB(u)}\right) = \frac{A^2(u) - uB(u)}{1 - uB(u)} \tag{16}$$

Introducing $F(u) = A(u) - 1$ (note that since $A(0) = 1$, $F(0)$=0), this becomes

$$F\left(\frac{uF(u)}{1 - uB(u)}\right) = \frac{F^2(u) + 2F(u)}{1 - uB(u)} \tag{17}$$

• We know that $F(0) = B(0) = 0$ and both $B$ and $F$ are continuous functions with B odd and F even so for small $u$, as a second order approximation we have

$$F(uF(u)) = 2F(u) \tag{18}$$

Since $F$ is continuous and $F(0) = 0$, $\forall\epsilon\ \exists\eta$ such that $|u| < \eta \Rightarrow |F(u)| < \epsilon$. In particular, we can choose $u_0 < \eta$ and $F(u_0) = F_0 = 2^{-n} < \epsilon$. Then we can set $u_1 = u_0F_0 = u_0 2^{-n}$ and $F_1 = F(u_1) = F(u_0F_0) = 2F_0 = 2^{1-n}$. Repeating this again, $u_2 = u_1F_1 = u_0 2^{-n}$ and $F_1 = F(u_1) = F(u_0F_0) = 2F_0 = 2^{1-n}$. This can be iterated as long as $F(u_p) < 1$ and one finds $F(u_p) = 2^{p-n}$. In particular $\exists p$ such that $F(u_p) \geq \frac{1}{2} > \epsilon$ while $u_p < u_0 < \eta$ contradicting the continuity hypothesis we started from unless $F(u) = 0$ and so $A(u) = 1$.

• With $A(u) = 1$, equation (13) becomes $vB(u) = uB(v)$ implying $B(v) = \kappa v$ where $\kappa$ is a constant to be identified.

• With this, equation (9) reads:

$$w = \frac{u+v}{1+\kappa vu} \tag{19}$$

and we recognize the form of the Lorentz velocity composition law.

• Equation (10) becomes

$$\gamma\left(\frac{u+v}{1+\kappa vu}\right) = \gamma(u)\gamma(v)(1+\kappa uv) \tag{20}$$

In the case $u = -v$ and using the fact that $\gamma(-v) = \gamma(v)$, this becomes $\gamma(0) = \gamma^2(v)(1-\kappa v^2)$. The case $v = 0$ then results in $\gamma(0) = \gamma^2(0)$ implying $\gamma(0) = 1$ and with this

$$\gamma(v) = \frac{1}{\sqrt{1-\kappa v^2}} \tag{21}$$

And we recover the Lorentz transformation with the identification $\kappa = \frac{1}{c^2}$

$$\begin{aligned} x' &= \gamma(u)(x - ut) \\ t' &= \gamma(u)(t - \kappa ux) \end{aligned} \tag{22}$$

## 3 Remarks on scaling factors

Any measurement of a physical quantity is accompanied with a scale (or resolution) to which the unit employed is generally connected. This is to say that only relative ratios of physical quantities can be accessed. With a length measurement in mind [3], any scale (resolution) ratio is usually considered to be equal to the ratio of scale ratios to a common reference scale according to:

$$\rho = \frac{\epsilon_2}{\epsilon_1} = \frac{\epsilon_2/\epsilon_0}{\epsilon_1/\epsilon_0} \tag{23}$$

which, in logarithmic form, takes the same form as a galilean velocity composition:

$$ln(\rho) = ln(\frac{\epsilon_2}{\epsilon_1}) = ln(\frac{\epsilon_2}{\epsilon_0}) - ln(\frac{\epsilon_1}{\epsilon_0}) \tag{24}$$

In fact, $\lambda_2$ and $\lambda_1$, the outcomes of two length measurements performed with different resolution in ratio $\rho$, are in the following relationship

$$\lambda_2 = \lambda_1/\rho^\delta \tag{25}$$

where $\delta$ is an always possible anomalous dimension. An example of interest is the case of Brownian motion for which the path length is characterized by $\delta = 1$. Using a common reference scale (resolution) $\lambda_0$, this can be written in a logarithmic form as:

$$ln(\lambda_2/\lambda_0) = ln(\lambda_1/\lambda_0) - \delta \cdot ln(\rho) \tag{26}$$

[3]It can be argued that any physical measurement can take the form of a position/length measurement via instrumentation and dial reading

In its logarithmic form, this relation is similar to a Galilean position coordinate transformation for position (one identifies $ln(\lambda_2/\lambda_0)$ to the position $x$, $\delta$ to time $t$ and $ln(\rho)$ to the relative velocity $v$). The underlying assumption that the anomalous dimension is independent on scale (resolution) provides the time coordinate transformation equivalent.

$$\delta_2 = \delta_1 \tag{27}$$

## 4 Concluding remarks

• In the derivation of the Lorentz transformation, the identification $\kappa = \frac{1}{c^2}$ is done after the fact and the three assumptions used were sufficient to imply a limit velocity.

• The only assumptions made for the derivation (internal composition and reflection invariance) can be seen as resulting from the principle of relativity ( the requirement that the equations of physics, have the same form in all reference frames.)

• The derivation of the Lorentz transformation suggests the galilean way we usually have to deal with scale (resolution) transformations is not justified. Some minimum as well as maximum scale (resolution) has to be considered. It is possible the minimum scale is 0 and the maximum scale is $\infty$ but this would be a particular case for which, there is no justification, just as there was no justification for the assumption implicitly made in classical mechanics that there was not limit velocity. The limit scales would be invariant under any dilation (change of scale or resolution) as the speed of light is invariant under any change of inertial reference system.

## References

[1] Einstein, A., Annalen der Physik 17: 891921, Zur Elektrodynamik bewegter Körper (On the Electrodynamics of Moving Bodies)

[2] Levy-Leblond J.M., 1976, Am. J. Phys. 44, 271, One more derivation of the Lorentz transformation.

[3] Nottale, L., 1992, Int. J. Mod. Phys. A7, 4899-4936, The theory of scale relativity