Lecture 6

Semi-classical motion in magnetic field

a) Free classical electron in magnetic field

\[ H = \frac{p^2}{2m}, \text{ coupling to the magnetic field} \]

\[ \mathcal{H} = \frac{1}{2m} (\mathbf{p} - e \mathbf{A})^2, \quad \mathbf{A} = \text{curl} \mathbf{A} \]

The equations of motion are

\[ \ddot{\mathbf{r}} = \frac{\partial \mathcal{H}}{\partial \mathbf{p}} = \frac{1}{m} \left( \mathbf{F} - \frac{e}{c} \mathbf{A} \right) = \frac{\mathbf{F}}{m} = \mathbf{F} \]

\[ \mathbf{p} = m \mathbf{v} \]

\[ \dot{\mathbf{p}} = -\frac{\partial \mathcal{H}}{\partial \mathbf{r}} = -\frac{1}{m} \mathbf{p} \mathbf{B} \cdot \left( -\frac{e}{c} \right) \frac{\partial \mathbf{A}}{\partial x} = -\frac{e}{c} \mathbf{v} \cdot \frac{\partial \mathbf{A}}{\partial x} \]

But

\[ \dot{\mathbf{p}} = \dot{\mathbf{p}}_x + e \frac{\partial \mathbf{A}}{\partial x} \cdot \mathbf{v} \]

\[ \mathbf{F} = \mathbf{p} = \mathbf{p}_x + e \frac{\partial \mathbf{A}}{\partial x} \cdot \mathbf{v} \]

\[ \rightarrow \quad \mathbf{F} = \frac{e}{c} \mathbf{v} \mathbf{p} \frac{\partial \mathbf{A}}{\partial x} - \frac{e}{c} \mathbf{v} \frac{\partial \mathbf{A}}{\partial x} \mathbf{p} = \]

\[ = \frac{e}{c} \mathbf{v} \mathbf{p} \left( \frac{\partial \mathbf{A}}{\partial x} - \frac{\partial \mathbf{A}}{\partial y} \right) \]

\[ \mathbf{F}_{\text{ps}} \sim \text{anti-symmetric tensor, } 3 \times 3 \]

\[ \mathbf{F}_{\text{ps}} \text{ has } 3 \text{ components } \quad \mathbf{T} = \begin{pmatrix} 0 & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{pmatrix} \]

and thus is "isomorphic" to a vector.
Levi-Civita symbol,

\[ \nabla \mathbf{A} = \varepsilon_{\alpha \beta \gamma} \partial_\gamma \mathbf{B} \]  - clearly antisymmetric, and has 3 independent components parameterized by \( \Theta_\alpha = 1 \ldots 3 \).

To extract \( \Theta_1 \), consider

\[ E_\alpha \xi_\beta \nabla_\beta = E_\alpha \xi_\beta \cdot \varepsilon_{\alpha \beta \gamma} \partial_\gamma \mathbf{B} = E_\alpha \xi_\beta \varepsilon_{\alpha \beta \gamma} \partial_\gamma \mathbf{B} = 2 E_\alpha \xi_\gamma \partial_\gamma \mathbf{B} = 2 E_\alpha \mathbf{A}_\alpha, \quad \Rightarrow \quad \text{switch dummy indices} \]

\[ \mathbf{B}_\gamma = \frac{1}{2} E_\alpha \varepsilon_{\alpha \beta \gamma} \left( \frac{\partial \mathbf{A}_\beta}{\partial x_\alpha} - \frac{\partial \mathbf{A}_\alpha}{\partial x_\beta} \right) = E_\alpha \partial_\beta \mathbf{A}_\alpha \]

\[ \Rightarrow \quad \mathbf{B}_\gamma = \varepsilon_\alpha \mathbf{A}_\alpha = \mathbf{B} - \mathbf{v} \times \mathbf{A} \quad \text{the magnetic field.} \]

Finally,

\[ \mathbf{p}_\alpha = E_\alpha \xi_\beta \cdot \nabla_\gamma \mathbf{B}_\gamma = \varepsilon_{\alpha \beta \gamma} \partial_\gamma \mathbf{B}_\gamma \mathbf{B}_\beta, \quad \text{or} \]

\[ \mathbf{p} = \frac{e}{2m} \mathbf{N} \times \mathbf{B} \quad \text{this is just the Lorentz force.} \]

The trajectory in the plane \( \perp \mathbf{B} \) is a circular.

Angular frequency:

\[ \omega_0 t = \frac{e}{c} \mathbf{v} \times \mathbf{B} \Rightarrow \omega_0 = \frac{eB}{mc} \]


\[ \text{the cyclotron frequency.} \]

Period:

\[ T = \frac{2\pi}{\omega_0} = \frac{2\pi mc}{eB} \]

The existence of classical periodic motion implies quantization of energy levels in \( \omega_0 \) - these are the celebrated Landau levels.
Classically, the radius of the orbit can be anything. In QM, there is a minimal radius ("the magnetic length"). To estimate we use
\[ m \omega R \cdot R = \frac{\hbar}{c} \] (or angular momentum \( \frac{\hbar}{c} \)).

Thus
\[ l_B^2 = \frac{\hbar^2}{m c} = \frac{\hbar c}{|B|} \]

Another way to remember: \( R_B^2 B = \frac{\hbar c}{2e} \).

Flux through \( l_B^2 \) is equal to the superconducting flux quantum (Because of \( 2e \) on the denominator.)

b) Bloch electron

When can we treat the motion quasi-classically?

Crudely speaking, the radius of the trajectory should be larger than the wave length of the particle,
\[ R_{\text{orbit}} \gg \lambda = \frac{\hbar}{|p|} \text{ (optical)} \]

Since \( R_{\text{orbit}} \ll l_B^2, l_B \gg \frac{\hbar}{|p|} \) ensures that the motion is always quasi-classical. We also could have demanded \( R_{\text{orbit}} \gg |e| B \), but \( l_B \gg R_{\text{orbit}} \) is the stronger condition, especially on superconductors.
where $E$ is the typical kinetic energy of an electron. In other words, $E$ should contain a lot of "quantum" of cyclotron energy on $e^2$.

Estimate for a semiconductor,

\[ E \approx E_F \approx \frac{\hbar^2}{2m} \Rightarrow m_e \approx 0.067 \text{ me} \]

\[ \frac{eB^2}{2m_e} \approx E_F \Rightarrow B \approx 100 \cdot 10^{-16} \cdot 0.067 \cdot 10^{-27} = 10^{-10} \]

\[ \approx 10^4 \text{ Tesla} \]

Thus we conclude that "It is a quantizing field for $E_F = 100$ and $m_e = 0.067 \text{ me}$". Indeed, typical QHE experiments are done in such fields.

In the semiclassical regime, we can write classical-like equations:

\[
\frac{\dot{p}}{p} = -\frac{\partial E_{\text{ext}}}{\partial p} \\
\frac{\dot{p}}{m_e} = -\frac{e}{m_e} \cdot \frac{\partial E_{\text{ext}}}{\partial p} \times \frac{1}{B} \text{ (cgs)}
\]

We just substituted $\frac{1}{m_e}$ with the bared velocity, $\frac{\partial E_{\text{ext}}}{\partial p}$. 
c) Trajectory in momentum space

To define the trajectory, it is useful to identify the constants of motion. Choose \( \vec{B} = (0, 0, B_z) \).

Hill's equation tells us that \( \dot{p}_2 = 0 \Rightarrow p_2 = \text{const} \). Further, \( \frac{\partial E}{\partial \vec{v}_{up}} = \frac{\partial E}{\partial \vec{p}} \Rightarrow \vec{p} \times \vec{B} = 0 \Rightarrow E_{up} = \text{const} \) (dink). These two integrals of motion, \( p_2 \) and \( E_{up} = E \), define the trajectory in the momentum space.

The intersection of the isoenergetic surface with the \( p_2 = \text{const} \) plane gives the trajectory in the momentum space. \( p_2 \) = const plane.

The sense of rotation must be chosen from \( \dot{e} = \frac{\vec{e}}{\nu_{up}} \), and \( \nu_{up} \) = \( \vec{v} \times \vec{B} \), and \( e < 0 \).

d) Trajectory in real space

\[ \vec{v} = \nu_{up} \Rightarrow -\frac{e}{c} \vec{B} \times \vec{v} = e \nu_{up} \times \vec{B} = \dot{p} \Rightarrow \]

\[ \Rightarrow -\frac{e}{c} \vec{B} \times d\vec{r} = dp \]

The trajectory in real space is the one in \( p \)-space rotated by \( 90^\circ \) in space and scaled.

\( (B \text{ is out of plane}) \)
\( \mathbf{p} = \mathbf{0} + \frac{2\epsilon \mathbf{e}_\parallel}{\gamma} \times \mathbf{B} \) \hspace{1cm} (2)

we can parameterize the motion along the trajectory with time, \( t \).

From (2) we get

\[
\frac{\mathrm{d} \mathbf{p}}{\mathrm{d} t} = \frac{1}{\gamma^1} \mathbf{B} \cdot \mathbf{v}_1 \mathbf{v}_2 \quad \text{modulus of the component of \( \mathbf{v} \) normal to \( \mathbf{B} \) vector}
\]

\( \mathbf{dp} \) - element of the trajectory traversed in time \( \mathrm{d} t \)

\[
\mathbf{dp} = \frac{1}{\gamma^1} \mathbf{B} \mathbf{v}_1 \mathrm{d} t \implies \int_{t_0}^{t} \mathbf{dp} = \frac{1}{\gamma^1} \mathbf{B} \mathbf{v}_1 \int_{t_0}^{t} \mathrm{d} t
\]

If the trajectory is closed, the period of motion is given by

\[
T = \frac{c}{\epsilon \mathbf{e}_\parallel} \int_{t_0}^{t} \frac{\mathrm{d} \mathbf{p}}{\mathbf{v}_1}
\]

This integral can be given a simple geometric interpretation:
Consider two nearby trajectories, corresponding to energies \( E \) and \( E + \Delta E \).

Since \( \nabla E = \frac{\partial E}{\partial \mathbf{r}} \) is the gradient of \( E \), it is \( \perp \) to the isoequeric surface \( S \).

Thus \( \frac{\Delta E}{\partial \mathbf{r}} \) is just the distance in momentum between the trajectories at every point. Thus if \( S(E) \) is the area inside the trajectory with energy \( E \), then

\[
S(E + \Delta E) = S(E) = \int dp \cdot \frac{\Delta p}{E} = \int \left[ \frac{dp}{E} = \frac{\partial S(E)}{\partial E} \right],
\]

and

\[
T = \frac{c}{\alpha \beta} \frac{\partial S}{\partial E}.
\]

Comparing to the free motion expr. \( T \approx \frac{\text{e}}{\beta \cdot \text{m}} \), we introduced the cyclotron mass,

\[
\mathbf{m}_c = \frac{1}{\beta \cdot \text{e}} \frac{\partial S}{\partial E}
\]

Check: \( E = \frac{p^2}{2m} \Rightarrow S = \pi p^2 = 2\pi i \text{m} E \Rightarrow \frac{\partial S}{\partial E} = \text{d}m \text{m}_c E \Rightarrow \)

\[
\mathbf{m}_c = \text{m}_c \mathbf{e}
\]
Semiclassical quantization in magnetic field.

Usually called Bohr- Sommerfeld quantization, but in the present context "Dirac quantization" is also used.

The semiclassical quantization rule is:

\[
\frac{1}{2m} \oint \frac{\mathbf{p} \cdot d\mathbf{r}}{c} = \hbar (n+\frac{1}{2}) \quad n \ll 1
\]

This quantization rule is a consequence of the canonical momentum

More explicitly,

\[
\frac{1}{\hbar} \oint P_x \cdot d\mathbf{r} = \frac{\theta}{\hbar} \quad \frac{1}{\hbar} \oint P_y \cdot d\mathbf{r} = n
\]

- \( \oint P_x \cdot d\mathbf{r} \): regular action/phase
- \( \oint P_y \cdot d\mathbf{r} = n \): phase accumulation under the magnetic field. \( \Phi_0 = \frac{\Phi}{\hbar c} \)

Aharonov-Bohm phase.

\[
\oint \mathbf{A} \cdot d\mathbf{r} = \frac{\Phi}{\hbar c} \quad \oint \mathbf{B} \cdot d\mathbf{r} = 2\pi \frac{\Phi}{\Phi_0} \quad \Phi_0 = \frac{\Phi}{\hbar c}
\]

\( \Phi_0 \) is the flux.

\( \Phi_0 \) is the flux.

NB: you may have heard about the "Berry phase" on a trajectory. When it is nonzero, the quantization rule is

\[
[\text{dynamic phase} + \text{AB phase} + \text{Berry phase}] = 2\pi (n+\frac{1}{2})
\]

(Berry phase can sometimes cancel that \( \frac{1}{2} \)).
Secular quantization applied to a regular 2DEG:

For instance:

\[ E = \frac{\mathbf{p}^2}{2m^*} \quad (\mathbf{p}^* = \mathbf{p} \cdot \mathbf{n}) \quad \text{or} \quad E = \frac{(\mathbf{p}_1 - \frac{e}{c} \mathbf{A})^2}{2m^*} + \frac{\mathbf{p}_2^2}{2m^*}. \]

Here the dispersion can be anything.

Choose the Landau gauge: \( \mathbf{A} = (-By, 0, 0), \quad \mathbf{B} = B_\perp \mathbf{e}_z \)

the Hamiltonian/energy does not depend on \( x \Rightarrow \)

the conjugate momentum, \( p_1 \), is conserved: \( p_1 = \text{const.} \)

\( \text{canonical!} \)

Thus

\[ \frac{1}{2m^*} \oint \mathbf{p} \cdot d\mathbf{r} = \frac{1}{2} \oint \left( \frac{\partial p_y}{\partial y} \mathbf{e}_y + \frac{\partial p_x}{\partial x} \mathbf{e}_x \right) \]

\[ = \frac{1}{2m^*} \oint \mathbf{e}_y dy + p_x \oint dx \]

\( \text{.c}, \text{as the motion is periodic.} \)

Since \( \frac{p_x}{m^*} = \mathbf{p}_x \cdot \mathbf{e}_x - \frac{e}{c} B_y \Rightarrow \mathbf{p}_x = \text{const.} \)

we conclude that \( dy = \frac{e}{EB} dp_x \quad (e < 0) \).

We see that

\[ \frac{1}{2m^*} \oint \mathbf{p} \cdot d\mathbf{r} = \frac{1}{2m^*} \oint \frac{p_y}{EB} \mathbf{e}_y dp_x = \]

\[ = \frac{1}{2m^*} \oint \mathbf{e}_y \cdot \mathbf{B} \cdot \mathbf{e}_z \cdot \mathbf{p} = \]

\[ = \frac{1}{2m^*} \oint \mathbf{B} \cdot \mathbf{p} \cdot \mathbf{e}_z = \]

Thus we get

\[ \frac{1}{2m^*} \frac{e}{EB} \oint \mathbf{p} \cdot \mathbf{e}_z = \hbar \left( \frac{h}{2} + \frac{1}{2} \right) \]

\( \Psi(E) = 2\pi \cdot \frac{eB}{c} \left( \frac{h}{2} + \frac{1}{2} \right). \)
For $E = \frac{p^2}{2m}$, \( S(E) = \pi p^2 = 2\pi \hbar \sqrt{E}$, and

we get

\[
\sqrt{\hbar m} E_n = \sqrt{\hbar} \frac{eB}{c} (n + \frac{1}{2}) \Rightarrow \left( E_n = \frac{\hbar}{\pi} \frac{eB}{mc} (n + \frac{1}{2}) \right) + \frac{\pi^2}{2m}
\]

these are the famous Landau levels.

**Semiclassical quantization for graphene**

In graphene low-energy electrons behave as massless particles: \( E = \sqrt{p}\)

The isoenergetic surfaces are circles,

\[
S(E) = \pi p(E) = \pi \frac{E}{\sqrt{E}}
\]

Further, the electrons are described by pseudospin, which points along quasimomentum. Thus when the momentum rotates by \( 2\pi \), so does the pseudospin, but the latter gives \( \pi \) in the wave function, or a phase shift of \( \pi \).

The quantization rule becomes. This an example of Berry phase.

\[
\frac{1}{\hbar} \oint \mathbf{P} d\mathbf{r} + \pi = 2\pi (n + \frac{1}{2}) \Rightarrow S(E) = 2\pi \hbar \frac{eB}{c} n \Rightarrow \left[ E = \pm \sqrt{2\hbar v^2 \frac{eB}{c} n} = \pm \frac{\hbar v}{eB} \sqrt{n} \right] \text{ Note a level at } h \approx 0!
\]

\[
\pi \frac{\pi^2}{2m} = 2\pi \hbar \frac{eB}{c} n \Rightarrow \left[ E = \pm \sqrt{2\hbar v^2 \frac{eB}{c} n} = \pm \frac{\hbar v}{eB} \sqrt{n} \right] \text{ exact expression!}
\]