Problem 1: $k$-dependence of the periodic part of a Bloch function

Prove that if $u_k$ does not depend on $k$, $\nabla_k u_k = 0$, then the periodic potential in which the particle moves is constant as a function of coordinates, that is $\nabla_r U(r) = 0$. It would be convenient for you to first prove that $\nabla_k u_k = 0$ implies $[H, \nabla_k H] = 0$.

Problem 2: Absence of band overlap in 1D

You might have noticed that the energy bands for 1D potentials were always drawn as non-overlapping. This is actually a general statement: there are only two states at every energy belonging to the same energy band in the 1D case (they are coupled by time-reversal symmetry, and correspond to $k$ and $-k$ quasimomenta). Prove this theorem. You have to think about the properties of a 1D Schrödinger equation as a differential equation to accomplish this.

Problem 3: Electrons in 1D periodic potential

Electrons move in a series of equidistant $\delta$-barriers, with distance between two neighboring ones equal to $a$. Each potential has the form of $V(x) = u\delta(x)$. This means that the total potential can be written as $V_{\text{tot}} = \sum_{n=-\infty}^{\infty} u\delta(x-na)$, $n$ are integers.

a) Assuming that the potential is weak (what is the criterion for its “weakness”?), calculate the value of the gap between the first and second energy bands. You may find it useful to review the “level repulsion” part of the lecture material, and think carefully what those two degenerate levels are which the periodic potential has to split to create a gap between the first and the second bands.

b) Without making any assumption about the strength of the potential, calculate the dispersion equation for the potential given above. The latter in this case is an implicit relation between the quasimomentum of the state (the $k$-vector), and its energy $\epsilon_k$. To find the dispersion equation recall that the Bloch theorem connects $\Psi(x)$ and $\Psi(x+a)$ (this introduces the $k$-vector), and you can obtain another relation between $\Psi(x)$ and $\Psi(x+a)$ from the standard boundary conditions across the $\delta$-potential.

c) Using your result from b), and applying the appropriate approximation, check that you recover the result for the first band gap from a).