Solutions for HW 3

Problem 1:

Let's do \((\cos \Theta \ e^{i \Phi})\), since the other case was considered in the lecture notes.

It is easy to see that \(A_\phi = 0\) and

\[
A_\theta = \imath \left( \cos \frac{\phi}{\theta} \ e^{i \Phi} \right) \left( \cos \frac{\phi}{\theta} \ e^{-i \Phi} \right) \left( \cos \frac{\phi}{\theta} \ e^{i \Phi} \right) = \cos^2 \theta
\]

Thus \(\Omega_{\phi \phi} = 0\), \(\Omega_{\phi \theta} = -\frac{1}{2} \sin \Theta\), as in the other gauge.

Problem 2:

One way to go is to express \(\Phi\) from the previous problem, and express \(\Theta, \Phi\) through \(h, \xi, \eta\), and calculate \(\Phi\) then.

Let's pretend we did not have that information.

Consider \(\Phi\) for definiteness: \(\Phi = (\Phi_1, \Phi_2)\),

\[
\begin{pmatrix}
\Phi_1 \\
\Phi_2
\end{pmatrix}
\begin{pmatrix}
\Phi_1 \\
\Phi_2
\end{pmatrix}
= h \begin{pmatrix}
\Phi_1 \\
\Phi_2
\end{pmatrix},
\text{thus}
\Phi_2 = h^2 \Phi_1, \quad h = \text{fix}.
\]

\(h \Phi_1 + h - \Phi_2 = \Phi_2, \quad h = h_x + i h_y \Rightarrow \Phi_2 = \frac{h - h_x}{h_y} \Phi_1\).
Normalization dictates
\[ (\Psi_1)^2 \left( 1 + \frac{(h - h_2)^2}{h_1 h_2} \right) = 1 \Rightarrow (\Psi_1)^2 = \frac{h^2 + h_2^2}{h^2 - 2 h h_2 + h_2^2 + h_1^2 h_2^2} = \frac{h^2}{2 h^2 - 2 h^2 h_2} = \frac{1}{2 h} \left( h + h_2 \right), \]
and we get
\[ |+\rangle = \sqrt{\frac{h + h_2}{2 h}} \left( \frac{1}{h - h_2} \right) \left( \frac{1}{\sqrt{2 h}} \right), \]
\[ |-\rangle \text{ can be obtained by time-reversal transform:} \]
\[ |-\rangle = -i \sigma_y |+\rangle \]
From here one can proceed by direct differentiation to get
\[ \Omega = -\frac{\hbar}{2} \frac{\Gamma_1}{h^3} \text{ (for } |+\rangle \text{ space)} \]

**Problem 3:**
\[ H = \hbar \mathbf{\hat{S}}, \text{ and we will use the general } S-L \]
\[ \Omega = -\imath \sum_{m+n} \langle N | \Theta^\dagger H | m \rangle \times \langle m | \Theta H | N \rangle \text{, where} \]
\[ (E_N - E_m)^2 \]
\[ |N\rangle : H |N\rangle = \hbar \mathbf{\hat{S}} |N\rangle. \]
We have \[ \Theta \mathbf{\hat{S}} = \mathbf{\hat{S}} = (S_x, S_y, S_z) = (\frac{S + S^-}{2}, \frac{S - S^+}{2i}, S_z) \]
Now we choose the instantaneous direction of \( \mathbf{L} \) as the quantization axis. Then the only matrix elements are for \( |m\rangle = |n\pm 1\rangle \).

\[
\langle n+1 | \mathbf{S} | n \rangle = \frac{1}{2} (1, - \epsilon, 0) \cdot \langle n+1 | \mathbf{S} + | n \rangle ,
\]

\[
\langle n-1 | \mathbf{S} | n \rangle = \frac{1}{2} (1, \epsilon, 0) \cdot \langle n-1 | \mathbf{S} - | n \rangle ,
\]

where

\[
\langle n+1 | \mathbf{S} + | n \rangle = \sqrt{S(S+1) - N(N+1)} \frac{E_n - E_{n+1}}{\hbar} ,
\]

\[
\langle n-1 | \mathbf{S} - | n \rangle = \sqrt{S(S+1) - N(N-1)} ,
\]

\[
\mathbf{\Omega} = -\text{Im} \left[ \frac{1}{4} \frac{(1, -i, 0) \times (1, i, 0)}{\hbar^2} \left( S(S+1) - N(N+1) \right) \right] -
\]

\[
-\text{Im} \left[ \frac{1}{4} \frac{(1, i, 0) \times (1, -i, 0)}{\hbar^2} \left( S(S+1) - N(N+1) \right) \right] =
\]

\[
= -\frac{\Theta_2}{2 \hbar^2} \left( S(S+1) - N(N+1) \right) + \frac{\hbar^2}{2 \hbar^2} \left( S(S+1) - N(N+1) \right) =
\]

\[
= -N \frac{1}{\hbar} \frac{\Theta_2}{2} ,
\]

Restoring the rotational invariance, we get

\[
\mathbf{\Omega} = -N \frac{1}{\hbar \sqrt{3}}
\]

- the field of a monopole, that goes back into \( \frac{\Theta_2}{2} = -\frac{1}{2} \frac{1}{\hbar} \) for \( N = \frac{1}{2} \).