Solutions for HW 4

Problem 1

The fact that any 2x2 matrix can be written as a linear combination of Pauli matrices can be seen as follows:

\[ M_{2 \times 2} = m_1 (0,0) + m_2 (0,1) + m_3 (1,0) + m_4 (1,1) \]

where \( m_i \) are all complex numbers. Now one can see explicitly that the basis matrices chosen above can be expressed via the Pauli matrices:

\[
\begin{align*}
(0,0) & = \frac{\sigma_0 + i \sigma_3}{2},
(0,1) & = \frac{\sigma_0 - i \sigma_3}{2},
(1,0) & = \frac{\sigma_1 + i \sigma_2}{2},
(1,1) & = \frac{\sigma_1 - i \sigma_2}{2}.
\end{align*}
\]

As a corollary, we can notice that only linear powers of Pauli matrices are independent, the higher ones can be expressed through them. This has a physical consequence: a spin \( \frac{1}{2} \) particle cannot have single spin anisotropy, like \( \text{Helm} = -g \mu_B \vec{B} \) - all such terms are really linear in spin operators.

Thus we know that

\[ F(a_0 + \bar{a} \sigma^0) = \omega + \frac{\bar{a} \omega}{2} \]

Let us consider \( a_0, \bar{a}, \) for which \( a_0 + \bar{a} \sigma^0 \) is Hermitian, and we can define \( \omega \) such that

\[ (a_0 + \bar{a} \sigma^0) |\psi\rangle = (a_0 + \omega) |\psi\rangle, \quad \omega^2 = \bar{a} \sigma^0, \]
It is clear (and can be proven) that $b = \beta \vec{a}$, since there is no other vector in the problem. Thus we have

$$F(a_0 + \alpha \vec{a}) = b_0 + \beta a \vec{a},$$

From

$$F(a_0 + \alpha \vec{a}) \times > = (b_0 + \beta a \vec{a}) \times > = F(a_0 + \alpha \vec{a}) \times >$$

we conclude that

$$b_0 + \beta a = F(a_0 + \alpha \vec{a}) \times >$$

$$b_0 - \beta a = F(a_0 - \alpha \vec{a}) \times >$$

and

$$F(a_0 + \alpha \vec{a}) = \frac{F(a_0 + \alpha \vec{a}) + F(a_0 - \alpha \vec{a})}{2} + \frac{F(a_0 + \alpha \vec{a}) - F(a_0 - \alpha \vec{a})}{2a} \vec{a}.$$

In the case of a spin-$\frac{1}{2}$ rotation operator, $U = e^{i\Theta \vec{\sigma}/2}$, we have $a_0 = 0$, $\vec{b} = \vec{\sigma}/2$, thus we get

$$U = \cos \frac{\Theta}{2} \vec{I} + \sin \frac{\Theta}{2} \vec{\sigma},$$

where $\vec{n} = \vec{\sigma}/\Theta$ is the direction of the rotation axis.
Problem 2

a) We have been using the Heisenberg picture in class, so let's use the Schrödinger one here. To find \( \frac{d}{dt} \langle \hat{S} \rangle \), we use \( \langle \hat{S} \rangle = \langle 413/4 \rangle \),

\[
\frac{d}{dt} \langle \hat{S} \rangle = \frac{i}{\hbar} \langle 413/4 \langle \hat{S} \rangle - \frac{1}{i} \langle 413\hat{S}/4 \rangle = \frac{1}{i} \langle 413[\hat{H}, \hat{S}]/4 \rangle;
\]

\[
[\hat{H}, \hat{S}_{\alpha}] = -g_{\mu B} [\hat{S}_{z}, \hat{S}_{\alpha}] = -g_{\mu B} \hat{S}_{\alpha} i\hbar E_{\parallel} Z 
\]

Thus we get

\[
\frac{d}{dt} \langle \hat{S} \rangle = \frac{i}{\hbar} (-g_{\mu B} + \chi) \langle \hat{S} \rangle \times \hat{B} = g_{\mu B} \cdot \hat{B} \times \langle \hat{S} \rangle
\]

By choosing the \( t \) axis along \( \hat{B} \), we get

\[
\frac{d}{dt} \langle \hat{S}_{z} \rangle = 0.
\]

\[
\frac{d}{dt} \langle \hat{S}_{x} \rangle = +g_{\mu B} \langle \hat{S}_{y} \rangle
\]

\[
\frac{d}{dt} \langle \hat{S}_{y} \rangle = -g_{\mu B} \langle \hat{S}_{x} \rangle
\]

b) The squares of moments found above do not include relaxation of the spin toward the equilibrium (along \( \hat{B} \)). Such terms cannot come from the Hamiltonian of
the spin only, we must add additional degrees of freedom that would serve as a "thermal bath" and provide re-balancing to the equation of motion phenomenologically. Such terms must not change the lambda of the spin and, only vectors we have to build a corresponding term one (B) and (B × S) would pull it to the equilibrium (see next fig). The"
The hopping Hamiltonian in question is

\[
\mathcal{H} = -\frac{t}{2} \sum_{\langle \langle \rangle \rangle} (|s_i^\uparrow s_j^\downarrow + h.c.) +
\]

\[
+ \sum_{\langle \langle \rangle \rangle} (-i\hat{\sigma}_x^\uparrow \hat{s}_i^\downarrow \hat{s}_j^\uparrow + i\hat{\sigma}_y^\uparrow \hat{s}_i^\uparrow \hat{s}_j^\uparrow + h.c.),
\]

where \( i,j \) label the sites of a square lattice with the lattice parameter \( a \).

(I added \( \frac{t}{2} \) in front of \( t \) to keep the usual hopping term in its usual form of \( -\frac{t}{2} \cos(k_a a) \).

By switching into Fourier space,

\[
|s_s^\uparrow > = \frac{1}{\sqrt{N}} \sum_{\mathbf{k}} e^{i \mathbf{k} \cdot \mathbf{x}} |\mathbf{k} s^\uparrow >,
\]

the Hamiltonian is brought into a diagonal form:

\[
\text{hopping term: } = \frac{t}{2} \sum_{\langle \langle \rangle \rangle} (|s_i^\uparrow s_j^\downarrow + h.c.) = \]

\[
= -\frac{t}{2} \left( \cos(k_a a) + \cos(k_b b) \right) = -2t \cos(k_a a + k_b b) \]

\[
\text{SOI term: }
\]

\[
\sum_{\langle \langle \rangle \rangle} i \hat{s}_i^\downarrow \hat{s}_j^\uparrow |s_i^\uparrow > <s_i^\uparrow | + h.c. =
\]

\[
= \frac{1}{N} \sum_{\mathbf{k}} \sum_{\langle \langle \rangle \rangle} e^{i \mathbf{k} \cdot \mathbf{x}} e^{i \mathbf{k} \cdot (\mathbf{e} + \mathbf{v})} |s_i^\uparrow > <s_i^\uparrow | + h.c.
\]
\[ \sum_k \xi_k s^x_s e^{-i \xi_k a} \psi_k = \sum_k \left( -i \tilde{a} s^x_s e^{-i \xi_k a} + i \tilde{a} s^x_s e^{i \xi_k a} \right) \psi_k \]

\[ = -2 \sum_k \sin \xi_k a \tilde{s}^x \psi_k \]

and the other piece is done analogously.

We can now read off the Hamiltonian in momentum space:

\[
H(k) = -2 + \cos \kappa x - 2 \cos \kappa y - 2 \sin \kappa y \sigma_x + 2 \sin \kappa y \sigma_y
\]

In the continuum limit, \( \kappa \ll 1 \), we obtain

\[ H(k) \approx \text{const} + a^2 \left( k_x^2 + k_y^2 \right) - 2 \alpha \kappa y \sigma_x + 2 \alpha \kappa y \sigma_y, \]

which coincides with the regular Rashba Hamiltonian with \( \alpha = 2 \kappa a \), \( m^* = \frac{1}{2a^2 \kappa} \).

\( k = \frac{1}{a} \)