Physics 3740 HW N° 8 Ch.7 : 30, 33, 36, 39, 40, 41, 42, 48, 58, 61, 63, 66, 67, 71, 73

(30)

7-30. Assuming the electron to be a classical particle, a sphere of radius 10^{-15} m and a uniform mass density, use the magnitude of the spin angular momentum $|S| = [s(s+1)]^{1/2}\hbar = (3/4)^{1/2}\hbar$ to compute the speed of rotation at the electron's equator. How does your result compare with the speed of light?

7-30. Angular momentum $S = I\omega = (2/5)mr^2(v/r)$ or

$$v = (5/2)S(1/mr) = 5S/2mr = 5(3/4)^{1/2}\hbar/2mr$$

$$= \frac{5(3/4)^{1/2}(1.055\times10^{-34}J\cdot s)}{2(9.11\times10^{-31}kg)(10^{-15}m)} = 2.51\times10^{11}m/s,$$ which is roughly $837c$!!

---

(33)

7-33. (a) The angular momentum of the yttrium atom in the ground state is characterized by the quantum number $j = \frac{3}{2}$. How many lines would you expect to see if you could do a Stern-Gerlach experiment with yttrium atoms? (b) How many lines would you expect to see if the beam consisted of atoms with zero spin, but $l = 1$?

7-33. (a) There should be 4 lines corresponding to the four m_j values $-3/2, -1/2, +1/2, +3/2$.

(b) There should be 3 lines corresponding to the three m_ℓ values $-1, 0, +1$.

---

(36)

7-36. A hydrogen atom is in the 3D state ($n = 3, l = 2$). (a) What are the possible values of j? (b) What are the possible values of the magnitude of the total angular momentum? (c) What are the possible z components of the total angular momentum?

7-36. (a) $j = \ell \pm 1/2 = 2 \pm 1/2 = 5/2$ *or* $3/2$

(b) $$J = \sqrt{j(j+1)}\,\hbar = \sqrt{\frac{5}{2}(5/2+1)}\,\hbar = 2.96\hbar$$

or $$= \sqrt{\frac{3}{2}(3/2+1)}\,\hbar = 1.94\hbar$$

(c) $\mathbf{J} = \mathbf{L} + \mathbf{S}$ and $J_z = L_z + S_z = m_\ell\hbar + m_s\hbar = m_j\hbar$ where $m_j = -j, -j+1, \dots j-1, j$. For $j = 5/2$ the z-components are $-5/2, -3/2, -1/2, +1/2, +3/2, +5/2$. For $j = 3/2$ the z-components are $-3/2, -1/2, +1/2, +3/2$. (all times $\hbar$).

---

(39)

7-39. Consider a system of two electrons, each with $l = 1$ and $s = \frac{1}{2}$. (a) Neglecting spin, what are the possible values of the quantum number for the total orbital angular momentum $\mathbf{L} = \mathbf{L}_1 + \mathbf{L}_2$? (b) What are the possible values of the quantum number S for the total spin $\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2$? (c) Using the results of parts (a) and (b), find the possible quantum numbers j for the combination $\mathbf{J} = \mathbf{L} + \mathbf{S}$. (d) What are the possible quantum numbers j_1 and j_2 for the total angular momentum of each particle? (e) Use the results of part (d) to calculate the possible values of j from the combinations of j_1 and j_2. Are these the same as in part (c)?

(P.T.O.)

(39) $l_1 = l_2 = 1$, $s_1 = \frac{1}{2} = s_2$ for each of the two electrons.

(a) $\vec{L} = \vec{L}_1 + \vec{L}_2$. Using the general rule on p. 315 for adding angular momentum quantum numbers:

$l \in \{ l_1 + l_2,\ l_1 + l_2 - 1,\ \ldots,\ |l_1 - l_2| \}$, or

$l \in \{ 1+1,\ 1+1-1,\ \ldots,\ |1-1| \}$, so $\boxed{l \in \{2, 1, 0\}}$.

(b) $\vec{S} = \vec{S}_1 + \vec{S}_2$.

$s \in \{ s_1 + s_2,\ s_1 + s_2 - 1,\ \ldots,\ |s_1 - s_2| \}$, or

$s \in \{ \frac{1}{2} + \frac{1}{2},\ \frac{1}{2} + \frac{1}{2} - 1,\ \ldots,\ |\frac{1}{2} - \frac{1}{2}| \}$, so $\boxed{s \in \{1, 0\}}$.

(c) $\vec{J} = \vec{L} + \vec{S}$.

$j \in \{ l+s,\ l+s-1,\ \ldots,\ |l-s| \}$.

Using the possible values of l and s found in (a) and (b) above we can make a table of values for j:

l	s	possible values of j
2	1	3, 2, 1
2	0	2
1	1	2, 1, 0
1	0	1
0	1	1
0	0	0

(d) $\vec{J}_1 = \vec{L}_1 + \vec{S}_1$, so $j_1 \in \{ l_1 + s_1,\ l_1 + s_1 - 1,\ \ldots,\ |l_1 - s_1| \}$

or $\boxed{j_1 \in \{ 1\frac{1}{2},\ \frac{1}{2} \}}$.

Similarly, $\boxed{j_2 \in \{ 1\frac{1}{2},\ \frac{1}{2} \}}$.

(P.T.O.)

39 (cont'd)

(e) $\vec{J} = \vec{J}_1 + \vec{J}_2$, so

$j \in \{ j_1 + j_2,\ j_1 + j_2 - 1, \ldots, |j_1 - j_2| \}$. Displaying the possible values of j_1 and j_2 [from part (d)] in a table, along with the corresponding possible values of j we find:

j_1	j_2	possible values of j
$1\frac{1}{2}$	$1\frac{1}{2}$	3, 2, 1, 0
$1\frac{1}{2}$	$\frac{1}{2}$	2, 1
$\frac{1}{2}$	$1\frac{1}{2}$	2, 1
$\frac{1}{2}$	$\frac{1}{2}$	1, 0

[Notice that as far as the values of j are concerned, the tables in parts (c) and (e) are identical: 3 appears once, 2 appears three times, 1 appears four times and 0 appears twice. Thus we see that the methods [(a) + (b) + (c)] and [(d) + (e)] are <u>equivalent</u> for the calculation of j. This is not really surprising, since vector addition is both commutative and associative, i.e.

$$\vec{J} = \vec{L} + \vec{S} = (\vec{L}_1 + \vec{L}_2) + (\vec{S}_1 + \vec{S}_2)$$
$$= (\vec{L}_1 + \vec{S}_1) + (\vec{L}_2 + \vec{S}_2)$$
$$= \vec{J}_1 + \vec{J}_2 .$$
]

(40) 7-40. The prominent yellow doublet lines in the spectrum of sodium result from transitions from the $3P_{3/2}$ and $3P_{1/2}$ states to the ground state. The wavelengths of these two lines are 589.6 nm and 589.0 nm. (*a*) Calculate the energies in eV of the photons corresponding to these wavelengths. (*b*) The difference in energy of these photons equals the difference in energy ΔE of the $3P_{3/2}$ and $3P_{1/2}$ states. This energy difference is due to the spin-orbit effect. Calculate ΔE. (*c*) If the $3p$ electron in sodium sees an internal magnetic field B, the spin-orbit energy splitting will be of the order of $\Delta E = 2\mu_B B$, where μ_B is the Bohr magneton. Estimate B from the energy difference ΔE found in part (*b*).

7-40. (a) $E_{3/2} = \dfrac{hc}{\lambda}$ Using values from Figure 7-22,

$$E_{3/2} = \frac{1239.852\ eV\cdot nm}{588.99\ nm} = 2.10505\ eV \qquad E_{1/2} = \frac{1239.852\ eV\cdot nm}{589.59\ nm} = 2.10291\ eV$$

(b) $\Delta E = E_{3/2} - E_{1/2} = 2.10505\ eV - 2.10291\ eV = 2.14\times 10^{-3}\ eV$

(c) $\Delta E = 2\mu_B B \Rightarrow B = \dfrac{\Delta E}{2\mu_B} = \dfrac{2.14\times10^{-3}\ eV}{2(5.79\times10^{-5}\ eV/T)} = 18.5\ T$

---

(41) 7-41. Show that the wave function of Equation 7-59 satisfies the Schrödinger equation (Equation 7-57) with $V = 0$ and find the energy of this state.

7-41. $\psi_{12} = \psi(x_1, x_2) = C\sin\dfrac{\pi x_1}{L}\sin\dfrac{2\pi x_2}{L}$. Substituting into Equation 7-57 with $V = 0$,

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2\psi_{12}}{\partial x_1^2} + \frac{\partial^2\psi_{12}}{\partial x_2^2}\right) = \left(\frac{\hbar^2}{2m}\right)(1+4)\left(\frac{\pi^2}{L^2}\right)\psi_{12} = E\psi_{12}$$

Obviously, ψ_{12} is a solution if $E = \dfrac{5\hbar^2\pi^2}{2mL^2}$.

---

(42) 7-42. Two neutrons are in an infinite square well with $L = 2.0$ fm. What is the minimum total energy that the system can have? (Neutrons, like electrons, have antisymmetric wave functions. Ignore spin.)

7-42. $\underbrace{E_n = \dfrac{n^2\hbar^2\pi^2}{2mL^2}}_{\text{(Eqn 6-24)}}$. Neutrons have antisymmetric wave functions, but if spin is ignored then one is in the $n = 1$ state, but the second is in the $n = 2$ state, so the minimum energy is:

$E = E_1 + E_2 = (1^2 + 2^2)E_1 = 5E_1$ where

$$E_1 = \frac{(\hbar c)^2\pi^2}{2mc^2L^2} = \frac{(197.3)^2\pi^2}{2(939.6)(2.0)^2} = 51.1\ MeV \Longrightarrow \quad E = 5E_1 = 255\ MeV.$$

(Here we have used $\hbar c = 197.3$ MeV-fm and $\underset{\text{(neutron)}}{mc^2} = 939.6$ MeV.)

48

7-48 If the 3*s* electron in sodium did not penetrate the inner core its energy would be $-13.6\text{ eV}/3^2 = -1.51$ eV. Because it does penetrate it sees a higher effective Z and its energy is lower. Use the measured ionization potential of 5.14 V to calculate Z_{eff} for the 3*s* electron in sodium.

7-48 $$E_n = -\frac{Z_{eff}^2 E_1}{n^2} \quad \text{(Equation 7-25)}$$

$$Z_{eff} = n\sqrt{\frac{-E_n}{E_1}} = 3\sqrt{\frac{5.14\,eV}{13.6\,eV}} = 1.84$$

58

7-58 The relative penetration of the inner-core electrons by the outer electron in sodium can be described by the calculation of Z_{eff} from $E = -[Z_{eff}^2(13.6\text{ eV})]/n^2$ and comparing with $E = -13.6\text{ eV}/n^2$ for no penetration (see Problem 7-48). (*a*) Find the energies in the outer electron in the 3*s*, 3*p*, and 3*d* states from Figure 7-22. (*Hint:* An accurate method is to use -5.14 eV for the ground state as given and find the energy of the 3*p* and 3*d* states from the photon energies of the indicated transitions.) (*b*) Find Z_{eff} for the 3*p* and 3*d* states. (*c*) Is the approximation $-13.6\text{ eV}/n^2$ good for any of these states?

7-58. (a) $\Delta E = hc/\lambda$

$$E(3P_{1/2}) - E(3S_{1/2}) = \frac{1240\,eV\cdot nm}{589.59\,nm} = 2.10\,eV$$

$$E(3P_{1/2}) = E(3S_{1/2}) + 2.10\,eV = -5.14\,eV + 2.10\,eV = -3.04\,eV$$

$$E(3D) - E(3P_{1/2}) = \frac{1240\,eV\cdot nm}{818.33\,nm} = 1.52\,eV$$

$$E(3D) = E(3P_{1/2}) + 1.52\,eV = -3.04\,eV + 1.52\,eV = -1.52\,eV$$

(b)

$$\text{For } 3P: \quad Z_{eff} = 3\sqrt{\frac{3.04\,eV}{13.6\,eV}} = 1.42$$

$$\text{For } 3D: \quad Z_{eff} = 3\sqrt{\frac{1.52\,eV}{13.6\,eV}} = 1.003$$

(c) The Bohr formula gives the energy of the 3D level quite well, but not the 3P level.

61

7-61. (*a*) Find the normal Zeeman energy shift $\Delta E = e\hbar B/2m_e$ for a magnetic field of strength $B = 0.05$ T. (*b*) Use the result of Problem 7-60 to calculate the wavelength changes for the singlet transition in mercury of wavelength $\lambda = 579.07$ nm. (*c*) If the smallest wavelength change that can be measured in a spectrometer is 0.01 nm, what is the strength of the magnetic field needed to observe the Zeeman effect in this transition?

7-61. (a) $\Delta E = \frac{e\hbar}{2m}B = \underbrace{(5.79\times10^{-5}\,eV/T)}_{\mu_B}\underbrace{(0.05\ T)}_{B} = 2.90\times10^{-6}\,eV$

(b) $|\Delta\lambda| = \frac{\lambda^2}{hc}\Delta E = \frac{(579.07\,nm)^2(2.90\times10^{-6}\,eV)}{1240\,eV\cdot nm} = 7.83\times10^{-4}\,nm$

(c) The smallest measurable wavelength change is larger than this by the ratio $0.01\text{ nm}/0.000783\text{ nm} = 12.8$ The magnetic field would need to be increased by this same factor since $B \propto \Delta E \propto \Delta\lambda$. The necessary field would be 0.639 T.

63

7-63. Show that the expectation value of r for the electron in the ground state of a one-electron atom is $\langle r\rangle = (3/2)a_0/Z$.

7-63. $\psi_{100} = \frac{1}{\sqrt{\pi}}\left(\frac{Z}{a_o}\right)^{3/2} e^{-Zr/a_o}$ (Equations 7-30 and 7-31)

$P(r) = 4\pi r^2\psi^*_{100}\psi_{100}$ (Equation 7-32)

$$= 4\pi r^2\frac{Z^3}{\pi a_o^3}e^{-2Zr/a_o} = \frac{4Z^3}{a_o^3}r^2e^{-2Zr/a_o}$$

$$\langle r\rangle = \int_0^\infty rP(r)dr = \int_0^\infty \frac{4Z^3}{a_0^3}r^3e^{-2Zr/a_0}dr$$

$$= \frac{a_o}{4Z}\int_0^\infty\left(\frac{2Zr}{a_o}\right)^3 e^{-2Zr/a_o}d(2Zr/a_o) = \frac{a_o}{4Z}\times3! = \frac{3a_o}{2Z}$$

(Recall: $\int_0^\infty x^n e^{-x}dx = n!$)

66

7-66. Find the minimum-value angle between the angular momentum **L** and the z axis for a general value of l, and show that for large values of l, $\theta_{min} \approx 1/l^{1/2}$.

7-66. $\theta_{min} = \cos^{-1}\left[m_\ell \hbar / \sqrt{\ell(\ell+1)}\,\hbar\right]$ with $m_\ell = \ell$. (Refer to Fig. 7-4, p. 277)

$\cos\theta_{min} = \ell/\sqrt{\ell(\ell+1)}$. Thus, $\cos^2\theta_{min} = \ell^2/[\ell(\ell+1)] = 1 - \sin^2\theta_{min}$

or, $$\sin^2\theta_{min} = 1 - \frac{\ell^2}{\ell(\ell+1)} = \frac{\ell(\ell+1)-\ell^2}{\ell(\ell+1)} = \frac{\ell^2+\ell-\ell^2}{\ell(\ell+1)} = \frac{\ell}{\ell(\ell+1)} = \frac{1}{\ell+1}.$$

And, $\sin\theta_{min} = \left(\frac{1}{\ell+1}\right)^{1/2}$ For large ℓ, θ_{min} is small. Then,

$$\sin\theta_{min} \approx \theta_{min} = \left(\frac{1}{\ell+1}\right)^{1/2} \approx \frac{1}{(\ell)^{1/2}}$$

67

7-67. The wavelengths of the photons emitted by potassium corresponding to transitions from the $4P_{3/2}$ and $4P_{1/2}$ states to the ground state are 766.41 and 769.90 nm. (*a*) Calculate the energies of these photons in electron volts. (*b*) The difference in energies of these photons equals the difference in energy ΔE between the $4P_{3/2}$ and $4P_{1/2}$ states in potassium. Calculate ΔE. (*c*) Estimate the magnetic field that the $4p$ electron in potassium experiences.

7-67. (a) $E_1 = hf = hc/\lambda_1 = 1240\,eV\cdot nm/766.41\,nm = 1.6179\,eV$

$E_2 = hf = hc/\lambda_2 = 1240\,eV\cdot nm/769.90\,nm = 1.6106\,eV$

(b) $\Delta E = E_1 - E_2 = 1.6179\,eV - 1.6106\,eV = 0.0073\,eV$

(c) As in Example 7-4, p. 317, $\Delta E \approx 2\mu_B B \Rightarrow B \approx \frac{\Delta E}{2\mu_B}$,

or $$B \approx \frac{0.0073\ eV}{2\left(5.79\times10^{-5}\ \frac{eV}{T}\right)} \approx \boxed{63\ T}.$$

71

7-71. Consider a hypothetical hydrogen atom in which the electron is replaced by a K^- particle. The K^- is a meson with spin 0, hence, no intrinsic magnetic moment. The only magnetic moment for this atom is that given by Equation 7-43. If this atom is placed in a magnetic field with $B_z = 1.0$ T, (*a*) what is the effect on the 1*s* and 2*p* states? (*b*) Into how many lines does the $2p \rightarrow 1s$ spectral line split? (*c*) What is the fractional separation $\Delta\lambda/\lambda$ between adjacent lines? (See Problem 7-60.) The mass of the K^0 is 497.7 MeV/c^2.

7-71. $\mu = -g_L \mu_B L/\hbar$ (Equation 7-43)

(a) The 1s state has $\ell = 0$, so it is unaffected by the external B.

The 2p state has $\ell = 1$, so it is split into 3 levels by the external B.

(b) The $2p \rightarrow 1s$ spectral line will be split into 3 lines by the external B.

71 (cont'd) (c) In Equation 7-43 we replace μ_B with $\mu_k = e\hbar/2\mu$, so

reduced mass: $\mu = \frac{m_k m_p}{m_k + m_p} = \frac{(497.7)(938.3)}{497.7+938.3}\ \frac{MeV}{c^2}$

$\Rightarrow \mu \simeq 325.2\ \frac{MeV}{c^2}$

$$\mu_{kz} = -(1)(1)(e\hbar/2\mu) = -\mu_B(m_e/\mu) \quad \text{(From Equation 7-45)}$$

Then $\Delta E = \mu_B(m_e/\mu)B$

$$= (5.79\times10^{-5}eV/T)\left[(0.511MeV/c^2)/(325.2MeV/c^2)\right](1.0T),$$

so $\Delta E \simeq 9.1\times10^{-8}\ eV$

$\frac{\Delta\lambda}{\lambda} = -\frac{\lambda}{hc}\Delta E$ (From Problem 7-60, where λ for the (unsplit) 2p $\rightarrow$ 1s transition is

given by $\lambda = hc/\Delta E_k$ and $\Delta E_k = E_2 - E_1 = -13.6eV\underbrace{(\mu/m_e)}_{\text{cf Eq (4-24)}}(1-1/4) = 6.49\times10^3\ eV$

and $\lambda = 1240eV\cdot nm/6.49\times10^3 eV = 0.191\ nm$

and $\frac{\Delta\lambda}{\lambda} = \frac{0.191nm(9.1\times10^{-8}eV)}{1240eV\cdot nm} = \boxed{1.4\times10^{-11}}$

---

73 **7-73.** In the anomalous Zeeman effect, the external magnetic field is much weaker than the internal field seen by the electron as a result of its orbital motion. In the vector model (Figure 7-29) the vectors **L** and **S** precess rapidly around **J** because of the internal field and **J** precesses slowly around the external field. The energy splitting is found by first calculating the component of the magnetic moment μ_J in the direction of **J** and then finding the component of **J** in the direction of **B**. (*a*) Show that $\mu_J = \frac{\boldsymbol{\mu}\cdot\mathbf{J}}{J}$ can be written

$$\mu_J = -\frac{\mu_B}{\hbar J}(L^2 + 2S^2 + 3\mathbf{S}\cdot\mathbf{L})$$

(*b*) From $J^2 = (\mathbf{L}+\mathbf{S})\cdot(\mathbf{L}+\mathbf{S})$ show that $\mathbf{S}\cdot\mathbf{L} = \frac{1}{2}(J^2 - L^2 - S^2)$. (*c*) Substitute your result in part (*b*) into that of part (*a*) to obtain

$$\mu_J = -\frac{\mu_B}{2\hbar J}(3J^2 + S^2 - L^2)$$

(*d*) Multiply your result by J_z/J to obtain

$$\mu_z = -\mu_B\left(1 + \frac{J^2 + S^2 - L^2}{2J^2}\right)\frac{J_z}{\hbar}$$

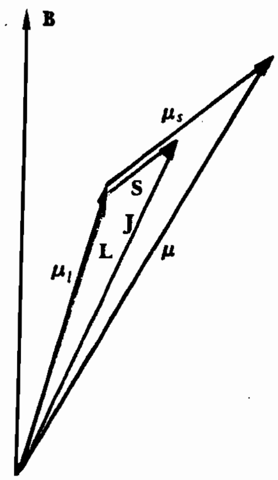

Figure 7-29
Vector diagram for the total magnetic moment when S is not zero. The moment is not parallel to the total angular momentum **J**, because μ_s/S is twice μ_l/L. (The directions of $\boldsymbol{\mu}_l$, $\boldsymbol{\mu}_s$, and $\boldsymbol{\mu}$ have been reversed in this drawing for greater clarity.)

7-73. (a) $\mathbf{J} = \mathbf{L} + \mathbf{S}$ $\qquad u = -\mu_B(L+2S)/\hbar$ (Equation 7-71)

$$u_J = \frac{u\cdot J}{J} = \frac{\left[-\mu_B(L+2S)/\hbar\right]\cdot[L+S]}{J}$$

$$= -\frac{\mu_B}{\hbar J}(L\cdot L + 2S\cdot S + 3S\cdot L)$$

$$= -\frac{\mu_B}{\hbar J}(L^2 + 2S^2 + 3S\cdot L).$$

(P.T.O.)

73 (cont'd) (b) $J^2 = J\cdot J = (L+S)\cdot(L+S) = L\cdot L + S\cdot S + 2S\cdot L \quad \therefore\ S\cdot L = \frac{1}{2}(J^2 - L^2 - S^2)$

(c) Substituting the result of (b) into the result of (a) gives

$$\mu_J = \frac{-\mu_B}{\hbar J}\left(L^2 + 2S^2 + 3\vec{S}\cdot\vec{L}\right)$$

$$= \frac{-\mu_B}{\hbar J}\left\{L^2 + 2S^2 + 3\left[\tfrac{1}{2}(J^2 - L^2 - S^2)\right]\right\}$$

$$= \frac{-\mu_B}{2\hbar J}\left(2L^2 + 4S^2 + 3J^2 - 3L^2 - 3S^2\right)$$

$$\Longrightarrow \boxed{\mu_J = \frac{-\mu_B}{2\hbar J}\left(3J^2 + S^2 - L^2\right)}. \qquad \text{Q.E.D.}$$

(d) The 'component of μ_J' in the z-direction [notice that this is not, in general, the same as the z-component of $\vec{\mu}$ (!)] is given by

$$\mu_z \equiv \mu_J \frac{J_z}{J} \overset{\text{[from (c)]}}{=} \frac{-\mu_B}{2\hbar J}\left(3J^2 + S^2 - L^2\right)\frac{J_z}{J}$$

(Note that $\frac{J_z}{J}$ is the cosine of the angle that $\vec{J}$ makes with the z-axis.)

$$= \frac{-\mu_B}{2J^2}\left(3J^2 + S^2 - L^2\right)\frac{J_z}{\hbar} = \frac{-\mu_B}{2J^2}\left(2J^2 + J^2 + S^2 - L^2\right)\frac{J_z}{\hbar},$$

or $$\boxed{\mu_z = -\mu_B\left(1 + \frac{J^2 + S^2 - L^2}{2J^2}\right)\frac{J_z}{\hbar}}. \qquad \text{Q.E.D.}$$

Using $J^2 = j(j+1)$, $S^2 = s(s+1)$, $L^2 = \ell(\ell+1)$, $\frac{J_z}{\hbar} = \frac{m_j\hbar}{\hbar} = m_j$

and $\Delta E = -\mu_z B$ (equation 7–69) we have, from (d) above:

$$\Delta E = \mu_B B\left\{1 + \frac{j(j+1) + s(s+1) - \ell(\ell+1)}{2j(j+1)}\right\} m_j, \quad \text{or}$$

$$\boxed{\Delta E = g\, m_j\, \mu_B B = g\, m_j\left(\frac{e\hbar B}{2m_e}\right)}$$ Eq. (7–72) … the energy level splitting

$$\boxed{g = 1 + \frac{j(j+1) + s(s+1) - \ell(\ell+1)}{2j(j+1)}}$$ Eq. (7–73) … the Landé g factor