Homework Assignment 6 Solution Set

PHYCS 4420

2 March, 2004

Problem 1 (Griffiths 4.26)

One way to find the energy is to find the $\vec{E}$ and $\vec{D}$ fields everywhere and then integrate the energy density for those fields. We know that $\vec{D}$ depends only on the free charge and is therefore continuous across the dielectric boundary with vacuum. Thus, the energy is

$$\begin{aligned} W &= \frac{1}{2}\int_{allspace} \vec{D}\cdot\vec{E} \\ &= \frac{1}{2}\int_a^b \frac{Q^2}{\epsilon(4\pi r^2)^2}d\tau + \frac{1}{2}\int_b^\infty \frac{Q^2}{\epsilon_0(4\pi r^2)^2}d\tau \\ &= \frac{Q^2}{8\pi\epsilon_0}\left(\frac{1}{1+\chi_e}\left(\frac{1}{a}-\frac{1}{b}\right)+\frac{1}{b}\right) \\ &= \frac{Q^2}{8\pi\epsilon_0}\frac{1}{1+\chi_e}\left(\frac{1}{a}+\frac{\chi_e}{b}\right). \end{aligned}$$

Problem 2 (Griffiths 4.32)

There is only one free charge: q. So inside and outside the dielectric we have

$$\vec{D} = \frac{q}{4\pi r^2}\hat{r}.$$

Inside the dielectric, then, we know

$$\vec{E} = \frac{q}{4\pi\epsilon_0(1+\chi_e)r^2}\hat{r}$$

and

$$\vec{P} = \epsilon_0\chi_e\vec{E} = \frac{\chi_e q}{4\pi(1+\chi_e)r^2}\hat{r}.$$

So the bound charges are

$$\begin{aligned} \sigma_b &= \vec{P}\cdot\hat{r}|_{r=R} = \frac{\chi_e q}{4\pi(1+\chi_e)R^2} \\ \rho_b &= -\vec{\nabla}\cdot\vec{P} = 0 \;\; \text{(except at } r=0\text{)}. \end{aligned}$$

The total bound charge on the surface is just $q\frac{\chi_e}{1+\chi_e}$, and it is compensated by the volume charge that is localized immediately around the implanted charge.

Note also that the field outside the dielectric is the same as if the dielectric were not present. The field is screened inside the dielectric but is otherwise unaffected once you are outside the dielectric again.

Problem 3 (Griffiths 4.33 - optional)

The behavior of $\vec{E}$ is known at the interface from the boundary conditions that arise due to the relationships between $\vec{E}$, $\vec{D}$ and the free and bound charges. We know that the parallel component of $\vec{E}$ should be continous across the boundary, but the perpendicular component has a discontinuity due to the surface charges there. However, we know that $\vec{D}$ only depends on *free* charges, so the perpendicular component of $\vec{D}$ is continuous across the interface. Thus,

$$\begin{aligned} \frac{\tan\theta_1}{\tan\theta_2} &= \frac{\frac{E_{1,\parallel}}{E_{1,\perp}}}{\frac{E_{2,\parallel}}{E_{2,\perp}}} \\ &= \frac{\frac{D_{2,\perp}}{\epsilon_2}}{\frac{D_{1,\perp}}{\epsilon_1}} \\ &= \frac{\epsilon_1}{\epsilon_2}. \end{aligned}$$

In the region $0 < \theta < \frac{\pi}{2}$, $\tan\theta$ is everywhere positive and grows with increasing θ. Thus, electric field lines behave opposite to the analogous light waves in media - they bend toward the normal when going from regions of large ϵ to small ϵ. Therefore, a convex lens would be a diverging lens for electric field lines (i.e., they would "defocus" the field).

Problem 4 (Griffiths 4.24 - optional)

This is a boundary value problem that is very similar to several others that we have done. This time, however, we add another boundary and a few more conditions based on the properties of dielectrics. The general solution is

$$V_i(r,\theta) = \sum_{l_i} \left(A_{l_i} r^{l_i} + \frac{B_{l_i}}{r^{l_i+1}} \right) P_{l_i}(\cos\theta)$$

where i denotes the region of interest ($i = 1, 2, 3$ for inside the conductor, inside the dielectric, and outside respectively) with the following boundary conditions:

$$\begin{aligned} V_3(r \to \infty) &\to -E_0 r\cos\theta && (1) \\ V_1 &= 0 \ \text{ (for simplicity)} && (2) \\ V_1(r=a) &= V_2(r=a) = 0 && (3) \\ V_2(r=b) &= V_3(r=b) && (4) \\ \epsilon \frac{\partial V_2}{\partial r}|_{r=b} &= \epsilon_0 \frac{\partial V_3}{\partial r}|_{r=b} && (5) \end{aligned}$$

From (1) we see that all A_{l_3} vanish except for $l_3 = 1$, for which $A_{1_3} = -E_0$. Condition (4) then tells us that

$$B_{l_3}\frac{1}{b^{l_3+1}} - B_{l_2}\frac{1}{b^{l_2+1}} - A_{l_2}b^{l_2} = 0 \ \ (\text{for } l \neq 0)$$

and

$$l_3 = l_2.$$

Thus,

$$(B_{l_3} - B_{l_2})\frac{1}{b^{l+1}} - A_{l_2}b^l = 0. \tag{6}$$

Now, condition (5) gives

$$\epsilon_r\left(\sum_{l_2}(l_2 A_{l_2} b^{l_2-1} - (l_2+1)B_{l_2}\frac{1}{b^{l_2+2}})P_{l_2}(\cos\theta)\right) = -E_0\cos\theta + \sum_{l_3} -(l_3+1)B_{l_3}\frac{1}{b^{l_3+2}}P_{l_3}(\cos\theta)$$

and so,

$$\frac{l+1}{b^{l+2}}(\epsilon_r B_{l_2} - B_{l_3}) - \epsilon_r l b^{l-1} A_{l_2} = 0 \ \ (\text{for } l \neq 0). \tag{7}$$

Now, take condition (3) which gives

$$A_{l_2} = -B_{l_2}\frac{1}{a^{2l+1}} \tag{8}$$

and combine it with results (6) & (7) above to get

$$B_{l_3} = B_{l_2}\left(1 + \frac{b^{2l+1}}{a^{2l+1}}\right) \tag{9}$$

$$B_{l_3} = B_{l_2}\epsilon_r\left(\frac{l+1}{b^{l+2}} + \frac{lb^{l-1}}{a^{2l+1}}\right) \tag{10}$$

$$\Longrightarrow B_{l_3} = B_{l_2} = A_{l_2} = 0 \ \ (\text{for } l \neq 0) \tag{11}$$

So, all that's left is to find A_{1_2} and B_{1_2} (we'll get B_{1_1} in the process). Go back to conditions (4) and (5) and write them again, this time with $l = 1$ (substituting for A via (8)). We have

$$-E_0 + B_{1_3}\frac{1}{b^3} = B_{1_2}\left(\frac{1}{b^3} - \frac{1}{a^3}\right) \tag{12}$$

$$E_0 + B_{1_3}\frac{2}{b^3} = B_{1_2}\epsilon_r\left(\frac{2}{b^3} + \frac{1}{a^3}\right) \tag{13}$$

$$\Longrightarrow B_{1_2} = \frac{-3E_0 a^3 b^3}{\epsilon_r(2a^3 + b^3) + 2b^3 - 2a^3} \tag{14}$$

$$\Longrightarrow A_{1_2} = -B_{1_2}\frac{1}{a^3} = \frac{3E_0 b^3}{\epsilon_r(2a^3 + b^3) + 2b^3 - 2a^3} \tag{15}$$

THEREFORE....

$$V_2(r,\theta) = \frac{3E_0 b^3}{\epsilon_r(2a^3+b^3)+2b^3-2a^3}\left(r-\frac{a^3}{r^2}\right)\cos\theta$$

and, so,

$$\begin{aligned}\vec{E}(\vec{r}) &= -\vec{\nabla}V(\vec{r})\\ &= -\frac{3E_0 b^3}{\epsilon_r(2a^3+b^3)+2b^3-2a^3}\left(\left(1+\frac{2a^3}{r^3}\right)\cos\theta\hat{r}-\left(1-\frac{a^3}{r^3}\right)\sin\theta\hat{\theta}\right).\end{aligned}$$

Apparently the screening by the dielectric makes it so that the electric field in the dielectric has some θ dependence as you move away from $r = a$. Thus, the bending of E_0 is weaker than if we had just placed the conducting sphere there without any dielectric.

Problem 5 (Griffiths 5.2)

We can just start from the result of Example 5.2

$$\begin{aligned}y(t) &= C_1\cos\omega t + C_2\sin\omega t + \frac{E}{B}t + C_3\\ z(t) &= C_2\cos\omega t - C_1\sin\omega t + C_4\end{aligned}$$

and apply the given initial conditions to solve for the unknown constants. In each case we have $x(0) = y(0) = z(0) = 0$ which tells us

$$\begin{aligned}C_1 &= -C_3\\ C_2 &= -C_4.\end{aligned}$$

a

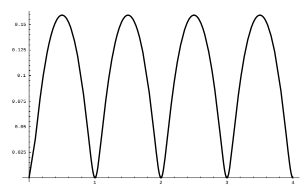
$$\begin{aligned}\frac{dy}{dt}|_{t=0} &= \frac{E}{B}\\ \frac{dz}{dt}|_{t=0} &= 0\\ &\Longrightarrow\\ C_2 &= 0\\ C_1 &= 0\\ &\Longrightarrow\\ y(t) &= \frac{E}{B}t\\ z(t) &= 0.\end{aligned}$$

In this case the magnetic and electric forces exactly cancel and the trajectory is a straight line in the $\hat{y}$ direction.

b

$$\begin{aligned}
\frac{dy}{dt}|_{t=0} &= \frac{E}{2B} \\
\frac{dz}{dt}|_{t=0} &= 0 \\
\Longrightarrow \\
C_2 &= -\frac{E}{\omega 2B} \\
C_1 &= 0 \\
\Longrightarrow \\
y(t) &= -\frac{E}{\omega 2B}\sin\omega t + \frac{E}{B}t \\
z(t) &= -\frac{E}{\omega 2B}\cos\omega t + \frac{E}{\omega 2B}.
\end{aligned}$$

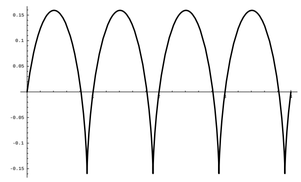
Now the velocity in the $\hat{y}$ direction is not enough to cause the magnetic force to cancel with the electric force, so there is some "rolling" like in Example 5.2. The trajectory is sketched below, with $\vec{z}$ as the verticle axis and $\vec{y}$ as the horizontal axis, both in units of $\frac{E}{B}$ with $\frac{Q}{m}$ such that $\omega = 2\pi sec^1$.



c

$$\begin{aligned}
\frac{dy}{dt}|_{t=0} &= \frac{E}{B} \\
\frac{dz}{dt}|_{t=0} &= \frac{E}{B} \\
\Longrightarrow \\
C_2 &= 0 \\
C_1 &= -\frac{E}{\omega B} \\
\Longrightarrow \\
y(t) &= -\frac{E}{\omega B}\cos\omega t + \frac{E}{B}t + \frac{E}{\omega B} \\
z(t) &= \frac{E}{\omega B}\sin\omega t.
\end{aligned}$$

This is another cycloid, but centered along the y axis. The sketch is below in the same units as the previous plot.



Problem 6 (Griffiths 5.6)

a

$$\vec{K}(r) = \sigma(r)\vec{v}(r) = \sigma\omega r\hat{\theta}$$

b

$$\vec{J}(r,\theta) = \rho\vec{v}(r,\theta) = \rho\omega r\sin\theta\hat{\phi}$$

Problem 7 (Griffiths 5.8)

a The field due to a finite straight segment of current is given in eq. 5.35 as

$$|\vec{B}(s)| = \frac{\mu_0 I}{2\pi s}(\sin\theta_2 - \sin\theta_1)$$

where s is the perpendicular distance from the current to the point and θ_2 andθ_1 are the angles to the ends of the current segment relative to the perpendicular s. The direction is determined by the right hand rule. At the center of a square loop the contribution to the magnetic field from each of the four sides points in the same direction (normal to the plane of the loop), so the total field at the center of a square of side $2R$ is just

$$\begin{aligned}\vec{B} &= \frac{\mu_0 I}{\pi R}\left(\sin\left(\frac{\pi}{4}\right) - \sin\left(-\frac{\pi}{4}\right)\right)\\ &= \frac{\sqrt{2}\mu_0 I}{\pi R}.\end{aligned}$$

b Generalizing to more sides just gives

$$\vec{B} = \frac{N\mu_0 I}{4\pi R}2\sin\frac{\pi}{N}$$

since $\sin\theta - \sin(-\theta) = 2\sin\theta$.

c As $N \to \infty$ we get

$$\lim_{N\to\infty} \frac{\mu_0 I}{2\pi R} \frac{N}{\pi} \sin\theta = \lim_{x\to 0} \frac{\mu_0 I}{2R} \frac{\sin x}{x} = \frac{\mu_0 I}{2R}.$$

This agrees with eq. 5.38, the field anywhere on the axis of a circular loop, when $z = 0$.

Problem 8 (Griffiths 5.13)

a From Ampere's law we get quite easily

$$\begin{aligned} \vec{B}_{inside} &= 0 \\ \vec{B}_{outside} &= \frac{\mu_0 I}{2\pi s}\hat{\theta} \end{aligned}$$

b

$$\begin{aligned} \vec{B}_{inside} &= \frac{\mu_0}{2\pi s} I_{encl} = \frac{\mu_0}{2\pi s} \int_0^s Cs' 2\pi s' ds' \\ &= \frac{C\mu_0 s^2}{3}\hat{\theta}, \end{aligned}$$

but how do we find C? We know that the total current is I, so

$$\begin{aligned} \int_0^a Cs' 2\pi s' ds' &= I \\ \implies C &= \frac{3}{2\pi a^3} I. \end{aligned}$$

This gives

$$\vec{B}_{inside} = \frac{\mu_0 s^2}{3} \frac{3}{2\pi a^3} I = \frac{\mu_0 I s^2}{2\pi a^3}\hat{\theta}$$

and, of course,

$$\vec{B}_{outside} = \frac{\mu_0 I}{2\pi s}\hat{\theta}.$$